Intersection Theory

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1 Rational Equivalence

For now, a variety for me is any **IRREDUCIBLE!!!** affine, quasi-affine, projective, or quasi-projective variety, cf. Hartshorne pg.15. Actually a variety is an integral reduced...???? scheme....????

Definition 1.1 (Hartshorne pg.16)**.** Let Y be a variety. We denote by $\mathcal{O}(Y)$ the ring of all regular functions on Y. If P is a point of Y, we define the *local ring of P on Y*, $\mathcal{O}_{P,Y}$ (or simply $\mathcal{O}_P$) to be the ring of germs of regular functions on Y, near P. In other words, an element of $\mathcal{O}_P$ is a pair $< U, f >$ where U is an open subset of Y containing P, and f is a regular function on U, and where we identify two such pairs (U, f) and (V, f) if $f = g$ on $U \cap V$.

It is important to know indeed $\mathcal{O}_P$ is a local ring, i.e. that it has a unique maximal ideal. Its maximal ideal $\mathfrak{m}$ is the set of germs/equivalence classes $[(U, g)]$ such that g vanishes at P. Indeed the complement of $\mathfrak{m}$ is the set of all units: if $f(P) \neq 0$, then $1/f$ is a regular function in some neighborhood of P.

Another fact about the maximal ideal is that the residue field $\mathcal{O}_P/k$ is isomorphic to the ground field k. This can be seen from the following short exact sequence:

$$0 \to \mathfrak{m} \hookrightarrow \mathcal{O}_P \xrightarrow{f(p)} k \to 0$$

Definition 1.2 (Hartshorne pg.15)**.** If Y is a variety, we define the *function field* $R(Y)$ of Y as follows: an element of $R(Y)$ is an equivalence class of of pairs (U, f) where U is a nonempty open subset of Y, f is a regular function on U, and where we identify two pairs (U, f) and (V, g) if $f = g$ on $U \cap V$. The elements of $R(Y)$ are called *rational functions* on Y.

Note that $R(Y)$ is indeed a field.

Proposition 1.3 (The Local Ring of a Subvariety. Hartshorne Excercise 3.13)**.** *Let $Y \subset X$ be a subvariety. Let $\mathcal{O}_{Y,X}$ be the set of equivalence classes $[(U, f)]$ where $U \subset X$ is open, $U \cap Y \neq \emptyset$, and f is a regular function on U. The equivalence relation is the following: $(U, f) \sim (V, g)$ is $f = g$ on $U \cap V$. Then $\mathcal{O}_{Y,X}$ is a local ring, with residue field $R(Y)$ and dimension* $= \dim X - \dim Y$. *It is called the* local ring of Y on X.

Proof. □

Note that this is just a generalization of the local ring at a point P: If $Y = P$ is a point we just get $\mathcal{O}_P$. Also, if $Y = X$ we get $R(X)$. Note also that if Y is not a point, then $R(Y)$ is not algebraically closed, thus in this way we get residue fields which are not algebraically closed.

Consider a variety X, and a subvariety V of codimension 1. Then the local ring $A = \mathcal{O}_{V,X}$ is a one-dimensional local domain. Let $r \in R(X)^*$ (i.e. $R(X) \setminus 0$, the multiplicative group), we will define the *order of vanishing of r along V* as the following:

$$\operatorname{ord}_V(r) := l_A \left(A \big/ (r) \right)$$

where l_A denotes the length of the A-module in parentheses.

Recall what the length of a module is:

Definition 1.4 (Length of a module)**.** Let M be a module over a ring R. Given a chain of submodules of the following form

$$N_0 \subsetneq N_1 \subsetneq \cdots \subsetneq N_n$$

we say that the chain has length n. The *length of the module* M is defined to be the supremum of the length of all of its chains.

1.1 Cycles and Rational Equivalence

Let X be an algebraic scheme. A k-cycle on X is a finite formal sum

$$\sum n_i[V_i]$$

where the V_i are k-dimensional subvarieties of X, and the n_i are integers.

We can put a group structure on the set of all k-cycles on X, this group is denoted Z_kX. The group operation is simply addition of finite linear combinations to get another finite linear combination. The additive identity is just 0 (times any subvariety?), and the inverses are the obvious ones. This group can also be described as the free abelian group on the k-dimensional subvarieties of X. To each subvariety V we can consider itself as an element of the group (just trivially with 1 as the coefficient), we will denote $[V]$ when considered as an element of the group.

For any $(k+1)$-dimensional subvariety W of X, and any $r \in R(W)^*$, define a k-cycle $[\mathrm{div}(r)]$ on X by

$$[\mathrm{div}(r)] := \sum \mathrm{ord}_V(r)[V]$$

the sum is taken over all codimension one subvarieties V of W.

A k-cycle $\alpha \in Z_kX$ is *rationally equivalent to zero*, written $\alpha \sim 0$, if there are a finite number of $(k+1)$-dimensional subvarieties W_i of X, and $r_i \in R(W_i)^*$, such that

$$\alpha = \sum [\mathrm{div}(r_i)]$$

The set of cycles rationally equivalent to zero form a subgroup $\mathrm{Rat}_k X$ of Z_kX. This can be readily observed from the fact that

$$[\mathrm{div}(r^{-1})] = -[\mathrm{div}(r)]$$

thus the inversion is closed in the subgroup.

The group of *k-cycles modulo rational equivalence*, or the *Chow group of k-dimensional cycles* is the quotient

$$A_kX = {Z_kX}\big/{\mathrm{Rat}_k X}$$

1.2 Pushforward of Cycles

First, we define the notion of a proper morphism of schemes.

Definition 1.5. Let $f : X \to Y$ be a morphism of schemes. Write

$$\Delta : X \to X \times_Y X$$

for the diagonal morphism, i.e. the natural morphism in the category of schemes

$$\Delta : X \xrightarrow{(Id,Id)} X \times X$$

We say that the morphism f is *separated* if $\Delta(X)$ is a closed subscheme of $X \times_Y X$, i.e. the diagonal map is a closed immersion.

We say that a scheme X is separated if the unique morphism

$$X \to \mathrm{Spec}(\mathbb{Z})$$

is separated.

This is an analogue of Hausdorffness. There is an equivalent definition of a topological space being Hausdorff if the diagonal $\Delta = \{(x,x) : x \in X\}$ is a closed subset of the product topological space $X \times X$.

Definition 1.6. Let $f : X \to Y$ be a morphism of schemes. We say that f is a *finite* morphism if Y has an open cover by affine shcemes $V_i = \operatorname{Spec} B_i$ such that for each i,

$$f^{-1}(V_i) = U_i$$

is an open affine subscheme $\operatorname{Spec} A_i$ (viewed as an open embedding), and the restriction of f to U_i, which induces a ring homomorphism

$$B_i \to A_i$$

makes A_i a finitely generated module over B_i

We say that f is a *finite type* morphism if

$$f^{-1}(V_i) = U_i$$

has a finite covering by affine open subschemes $U_{ij} = \operatorname{Spec} B_{ij}$ with B_{ij} being an A_i-algebra of finite type (i.e. finitely generated as an A_i algebra).

Definition 1.7. Let $f : X \to Y$ be a morphism of schemes. We say that f is *universally closed* if for every scheme Z with a morphism $Z \to Y$, the projection from the fiber product

$$X \times_Y Z \to Z$$

is a closed map of the underlying topological spaces.

And finally,

Definition 1.8. Let $f : X \to Y$ be a morphism of schemes. We say that f is a *proper* morphism if it is separated, of finite type, and universally closed.

We need the following fact which we will not prove right now:

Proposition 1.9. *Let $f : X \to Y$ be a proper morphism. Then for any subvariety V of X, the image $W = f(V)$ is a closed subvariety of Y.*

In such a case (proper morphism), we get an imbedding of fields

$$R(W) \hookrightarrow R(V)$$

To justify this, first consider a dominant morphism of varieties $h : A \to B$, i.e. a morphism in which the image is dense (this is trivially true in our case from V to $W = f(V)$). Suppose $\phi \in R(B)$ is a rational function on B, thus by definition it is an equivalence class $[(U, g \in \mathcal{O}(U))]$, under the familiar equivalence relation. Pick a representative (U, g) for ϕ. Since $f(A)$ is dense, $f^{-1}(U)$ is non-empty. Hence $[(f^{-1}(U), f \circ g)]$ is a rational function on X. We see that equivalent functions pullback to equivalent function. In this way we obtain an embedding $R(B) \hookrightarrow R(A)$.

Returning to the case of Proposition 1.9, it is a fact from EGA that $R(V)/R(W)$ is a finite field extension if W has the same dimension as V. Thus we can set

$$\deg(V/W) := \begin{cases} [R(V) : R(W)] & \text{if } \dim(W) = \dim(V) \\ 0 & \text{if } \dim(W) < \dim(V) \end{cases}$$

uppose V is k-dimensional in X, we can also define a k-cycle in Y to be

$$f_*[V] = \deg(V/W)[W]$$

We can extend linearly to define a ring homomorphism, called the *push-forward*

$$f_* : Z_k X \to Z_k Y$$

Theorem 1.10. *If $f : X \to Y$ is a proper morphism, and α is a k-cycle on X which is rationally equivalent to zero, then $f_*\alpha$ is rationally equivalent to zero on Y.*

2 Flat Pullback of Cycles

Definition 2.1. A morphism of schemes $f : X \to Y$ is *flat* if the induced morphism on stalks at every $P \in X$:

$$f_P^{\#} : \mathcal{O}_{Y,f(P)} \to \mathcal{O}_{X,P}$$

is a flat morphism of rings, i.e. this morphism makes $\mathcal{O}_{X,P}$ a flat $\mathcal{O}_{Y,f(P)}$-module. (Recall that a morphism of schemes has an underlying morphism of sheaves on Y

$$f^{\#} : \mathcal{O}_Y \to f_* \mathcal{O}_X$$

which induces a local ring homomorphism of stalks

$$f_P^{\#} : \mathcal{O}_{Y,f(P)} \to \mathcal{O}_{X,P}$$

in which the locality is required as part of the definition of the morphism of a locally ringed space.)

for interpretation, see `https://mathoverflow.net/questions/6789/why-are-flat-morphisms-flat`

3 Cycles of Schemes

4 Flat Pullbacks of Cycles

Definition 4.1. A morphism of schemes $f : X \to Y$ has relative dimension n if for all subvarieties V of Y, and all irreducible components V' of $f^{-1}(V)$, $\dim V' = \dim V + n$.

Fact: If f is flat, Y is irreducible, and X has pure dimension equal to $\dim Y + n$, then f has relative dimension n, and all base extensions $X \times_Y Y' \to Y'$ have relative dimesion n.

We assume every flat morphism to have a relative dimension n for some integer n.

For a flat morphism $f : X \to Y$, and any subvariety V of Y,set

$$f^*[V] = [f^{-1}(V)]$$

where $f^{-1}(V)$ is the inverse image scheme with scheme structure given by fiber products, which is a subscheme of X of pure dimension $\dim(V) + n$ (from flatness), and $[f^{-1}(V)]$ is this subscheme's cycle (of schemes). We can extend by linearity to *flat pullback homomorphisms* (of rings)

$$f^* : Z_k Y \to Z_{k+n} X$$

Lemma 4.2. *If $f : X \to Y$ is flat, then for any subscheme Z of Y,*

$$f^*[Z] = [f^{-1}(Z)]$$

Theorem 4.3.