Introduction to Lie Groups

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1 Topological Groups

Given a group G, we denote the identity, or neutral element by $e \in G$. The group induces two maps, the "multiplication map":

$$\begin{aligned} m\colon G \times G &\to G \\ (x, y) &\mapsto x \cdot y \end{aligned}$$

And the "inversion map":

$$\begin{aligned} i\colon G &\to G \\ x &\mapsto x^{-1} \end{aligned}$$

Definition 1.1. Let G be a group. A topology $\tau \subset \mathcal{P}(G)$ endows G with the structure of a *topological group* if the multiplication map and the inversion map are both continuous (with respect to the topology).

There are properties of topological groups we can immediately talk about.

Proposition 1.2. *The inversion map i is continuous and bijective. Since its inverse i^{-1} is also continuous, i is in fact a homeomorphism.*

Proof. Trivial. □

Proposition 1.3. *Fix an element $g \in G$. Define the* left-translation by g *as the mapping:*

$$\begin{aligned} L_g\colon G &\to G \\ x &\mapsto gx \end{aligned}$$

Then L_g is continuous and bijective; and its inverse is $L_{g^{-1}}$, which is also continuous, therefore L_g is a homeomorphism.

Proof. Continuity follows from Definition 1.1. The fact that L_g is a bijection follows from the group axioms. We check that $L_{g^{-1}}$ is indeed the inverse:

$$(L_{g^{-1}} \circ L_g)(x) = L_{g^{-1}}(gx) = g^{-1}gx = x$$

□

Proposition 1.3, together with associativity of multiplication, indicates that a topological group "looks the same locally everywhere". This is because whatever happens near a point, we can homeomorphically translate to the neutral element by the inverse, then homeomorphicaly translate to any other point. In the same spirit, whatever happens near the neutral element happes everywhere else. This will turn out to be enormously useful for us.

Proposition 1.4. *Let G be a topological group. Let $H \leq G$ be a subgroup. Then H is a topological group when endowed with the subspace topology.*

Proof. □

We now give some examples of topological groups.

Example 1.5. The group $(\mathbb{R}^n, +)$ equipped with the Euclidean topology.

Example 1.6. The multiplicative groups $(\mathbb{R}^*, \cdot)$ and $(\mathbb{C}^*, \cdot)$ equipped with the topology induced by the Euclidean topology, when regarded as a subspace of $\mathbb{R}$ and $\mathbb{C}$ respectively.

Example 1.7. Let $M_{n,m}(\mathbb{R})$ denote the vector space of $n \times m$ matrices with entries in $\mathbb{R}$. Then if we identify $M_{n,m}(\mathbb{R})$ with $\mathbb{R}^{nm}$ by just considering every matrix to be an $n \times m$ tuple, we get a natural topology on $M_{n,m}(\mathbb{R})$. Now We can look at the subspace

$$\mathrm{GL}(n, \mathbb{R}) \coloneqq \{A \in M_{n,m}(\mathbb{R}) : \det(A) \neq 0\} \subset M_{n,m}(\mathbb{R})$$

We know from linear algebra that $\mathrm{GL}(n, \mathbb{R})$ is in fact a group with matrix multiplication as the group operation,, and identiy element Id_n the $n \times n$ identity matrix. So if we endow $\mathrm{GL}(n, \mathbb{R})$ with the subspace topology from $M_{n,m}(\mathbb{R})$, we can see that $\mathrm{GL}(n, \mathbb{R})$ is a topological group. We verify here that the multiplication map and the inversion map are indeed continuous:

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