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Kontsevich's Formula for Rational Plane Curves, Gromov-Witten Invariants, and Quantum Cohomology

Semesterarbeit

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Spring 2019

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Chapter 0

Introduction

The primary purpose of this article is to give two proofs of the following fact:

Let N_d be the number of smooth projective rational plane curves passing through $3d-1$ points. Then the following recursive formula holds:

$$N_d = \sum_{d_A+d_B=d} N_{d_A} N_{d_B} d_A^2 d_B \left(d_B \binom{3d-4}{3d_A-1} - d_A \binom{3d-4}{3d_A-1} \right)$$

Starting with the trivial fact $N_1 = 1$, this formula allows one to compute N_d for any d.

This formula was originally proven by Kontsevich and Manin in their 1994 paper [11] using the then new methods involving stable maps, quatum cohomology, and Gromov-Witten invariants. Central to all of these was the role played by the moduli space $\overline{M}_{0,n}(\mathbb{P}^r, d)$ classifying morphisms $C \to \mathbb{P}^r$ from an n-pointed at-worse nodal projective rational curve C to $\mathbb{P}^r$ of degree d, subject to a certain stability condition, which is equivalent to having finite automorphisms.

We will postpone the proof of Kontsevich and Manin after a more elementary direct proof is given, which is done via counting maps in $\overline{M}_{0,n}(\mathbb{P}^r, d)$ meeting prescribed conditions using essentially elementary combinatorics to justify the terms in the formula.

Then, we will introduce (genus-0) Gromov-Witten invariants on $\overline{M}_{0,n}(\mathbb{P}^r, d)$, which essentially enumerates maps that meet these prescribed conditions, and whose generating functions we use to define the quantum product on the cohomology of $\mathbb{P}^r$. The associativity of this quantum product turns out to be equivalent to the Witten-Dijkgraaf-Verlinde-Verlinde (WDVV) differential equations, which in turn gives the above recursive formula in a special case.

The road to both proofs is paved in the language of *moduli spaces*. Specifically, moduli spaces of genus-0, nodal pointed curves, and moduli spaces of maps from these curves to $\mathbb{P}^r$. Moduli spaces can be thought of as the algebro-geometric version of a *classifying space*. The precise definition of such a space in an algebro-geometric setting requires a suitable definition for a *family* of objects to classify, and a universal property satisfied by the moduli space, as the unique base parametrizing some universal family.

This article is expository in nature. Throughout, we follow closely the book *An Invitation to Quantum Cohomology* by Joachim Kock and Israel Vainsencher, and *Notes on Stable Maps and Quantum Cohomology* by William Fulton and Rahul Pandharipande (which we denote by the short hand F-P). We (definitely) do not claim any new results in this article.

Outline

We start in Chapter 1. with an introduction and overview of the notion of a moduli problem, and conditions on an algbebro-geometric object to be a fine or course moduli space for a moduli problem. The spirit of the moduli problem is in asking for the best algebro-geometric object (varieties, schemes, stacks,...) whose geometry reflects how isomorphism classes vary. In other words, the most natural parametrization of the classes by some algebro-geometric object. As with most statements involving the word *best* or *natural*, the precise statement is a universal property on families of objects.

Subsequently, the bulk of the paper of Chapters 2. and 3. consists of working towards a description of $\overline{M}_{0,n}(\mathbb{P}^r, d)$. This requires a trek through several other moduli problems and moduli spaces: the moduli space $M_{0,n}$ classifying n-pointed smooth rational curves, and its Delign-Mumford-Knudsen compactification $\overline{M}_{0,n}$ which adds in nodal curves; and the moduli space $M_{0,n}(\mathbb{P}^r, d)$ which classifies degree d maps from an n-pointed smooth rational curve C to $\mathbb{P}^r$, of which $\overline{M}_{0,n}(\mathbb{P}^r, d)$ will be a compactification of. In each of these, our focus will be on the precise formulation of the moduli problem associated to it. We give the constructions of $M_{0,n}$ and $M_{0,n}(\mathbb{P}^r, d)$, and prove they are indeed (fine) moduli spaces for their associated moduli problems. However, the corresponding results for their compactifications are omitted for they are beyond the scope of this article.

In Chapter 4. we give a direct proof of the formula by utilizing a combinatorial relation on the boundary $\overline{M}_{0,n}(\mathbb{P}^r, d) \setminus M_{0,n}(\mathbb{P}^r, d)$.

In Chapter 5, we introduce genus-0 Gromov-Witten invariants, which roughly speaking, counts the number of curves in $\mathbb{P}^r$ meeting prescribed incidence conditions.

In Chapter 6, we introduce the quantum cup product, a binary operation defined on the cohomology groups of $\mathbb{P}^r$ (and more generally for an arbitrary homogeneous variety X), which are defined in terms of partial derivatives of generating functions of the Gromov-Witten invariants. Subsequently, we show that the associativity of said product is equivalent to the recursive formula above.

Background Knowledge

We assume a working knowledge of basic algebraic geometry roughly at the level of Chapters 1., 2., and some parts of 3. of Hartshorne [7].

Some knowledge of intersection theory will be needed, mainly the contents of Chapters 1. and 2. from the book by Fulton [5]. See the Appendix A for a selection of important topics.

Some elementary complex analysis will also be assumed. We will freely use the theory of complex manifolds and analytic spaces when needed. In particular we use some results concerning genus-0 Riemann surfaces and complex algebraic curves.

Knowledge of the theories of singular homology and cohomology from algebraic topology is also needed.

Conventions and Notations

We will work over $\mathbb{C}$. Thus, a *scheme* is a scheme over $\mathbb{C}$. We use the notation $\bullet := \mathrm{Spec}(\mathbb{C})$.

If unspecified, fiber products are taken over $\mathrm{Spec}(\mathbb{C})$, i.e. if

$$X \to \mathrm{Spec}(\mathbb{C}), \quad Y \to \mathrm{Spec}(\mathbb{C})$$

are schemes over $\mathbb{C}$, then

$$X \times Y \text{ means } X \times_{\mathrm{Spec}(\mathbb{C})} Y$$

We will denote the projection maps associated to the fiber product $X \times_S Y$ by $p_Y : X \times_S Y \to Y$ and $p_X : X \times_S Y \to X$.

Since all schemes will be assumed to be over $\mathbb{C}$, the set of *closed points* and $\mathbb{C}$-*valued points* or *geometric points* of a scheme are in one-to-one correspondence and thus will be used interchangeably. In particular a *geometric fiber* will mean a fiber over such a point.

A *variety* is a reduced, integral scheme of finite type over $\mathbb{C}$. Equivalently, it is a quasi-projective variety over $\mathbb{C}$ from the viewpoint of classical algebraic geometry. In particular, a variety is irreducible.

A *curve* is a variety of dimension one.

We will use the terms *map* and *morphism* interchangeably.

Acknowledgements

I express, first and foremost, my gratitude to Professor Dr. Rahul Pandharipande for giving me the opportunity to conduct this semester project and for his constant support and encouragement for my work.

I thank Dr. Longting Wu for his guidance and supervision; his willingness to accommodate my request for explanation and guidance throughout this process is the reason I have learned so much from this project.

I also thank Dr. Honglu Fan for invaluable discussions. His ability to explain difficult and impenetrable ideas with comforting and enlightening concrete examples was a great source of encouragement.

Discussions with E. Batelier, M. Cely, L. Chen, V. Chen, D. Choo, R. De los Santos, M. Duany, K. Drageset, G. Felber, F. Fournier, J. Glas, W. Hornslien, J-T. Lin, M. McClimans, T. Meyer, A. Mugnier, A. Musso, V. Nusbaumer, C. Pütz, K. Snyder, N. Stemen, D. Xiao, S. Zhang were of great value and provided insights and motivation for this project.

Finally, I thank my comrade colleagues — students of the Eidgenössische Technische Hochschule Zürich for their invaluable and indispensable presence in motivating, helping, encouraging, and inspiring me to do good mathematics.

Chapter 1

Moduli Problems and Moduli Spaces

1.1 The Classification Problem

A *classification problem* is the task of describing a class of objects up to a chosen equivalence relation. For us, the objects are algebro-geometric objects, although in a possibly more general setting they need not be. In other words, given a collection S of objects (schemes, curves, rational curves, varieties,...), and an equivalence relation $\sim_S$ on S, we wish to describe the set of equivalence classes $S/\sim_S$. For algebraic geometers, "to describe" amounts to asking if there is a natural scheme, or possibly some more general algebro-geometric structure, on $S/\sim_S$. This approach to a classification problem we will term a *moduli problem*.

There are more than one set of criteria of varying strength for the suitability and naturality of an algebro-geometric structure on $S/\sim_S$. One of the requirements we would like the structure to have is that it "parametrizes" the isomorphism classes in the best way possible, which we will define in terms of a universal property. In algebraic geometry, the notion of a collection of schemes parametrized by some fixed base scheme is formalized by viewing a morphism $S \to B$ as a collection of fibers parametrized by the closed points of B, a.k.a. a *family over/parametrized by B*. We will however also need to impose conditions and structures on the morphism to suit the types of objects we want to classify at hand.

In the best case scenario, to find the most suitable scheme structure will amount to finding such a morphism, in which each fiber is isomorphic to (a representative of) an isomorphism class of objects. The base scheme will be called the *(fine) moduli space*, and is what we are looking for.

However, in many cases, many classes of objects we would naturally want to classify do not yield the existence of such a morphism and a corresponding base scheme. Thus in the theory of moduli we frequently relax the conditions of the universal property to get the next-best thing. We will introduce this notion as the *coarse moduli space*.

In more sophisticated and advanced circles, even a coarse moduli space may not exist for some particular moduli problem. It may even be the case that no suitable scheme structure can be found. The study of algebraic stacks was largely motivated by a need to enlarge the category of schemes to find suitable moduli spaces in these scenarios. We will not discuss anything related to algebraic stacks in this paper.

Even though it is possible in the general theory of moduli to consider general types of objects and general types of mathematical structures, for the purpose of this paper, our objects in S are always some particular collection of schemes or morphisms of schemes, possibly with some extra structures; the equivalence relation on the objects of S will always be scheme isomorphism, or isomorphism of morphisms, respectively; and whatever extra strcutures must be respected by the isomorphism. For example, we may consider the case of classifying rational curves up to isomorphism, in which case $S = \{\text{smooth rational curves}\}$, and $\sim_S$ is scheme isomorphism (equivalently, in this case, isomophism of varieties); or we may consider the case of classifying morphisms $C \to \mathbb{P}^r$ from a smooth rational curve C, up to equivalence of maps, i.e. maps are considered equivalent if there is an isomorphism on the source curves respecting the individual maps. We will make these notions precise as we go.

1.2 Families of Objects, Pullbacks, and the Moduli Functor

We will first need the notion of a *family* of objects. Intuitively, it is a collection of objects parametrized or indexed by the geometric points of a *base scheme* B. This is formalized by the notion of a morphism of schemes:

$$\begin{array}{c} X \\ \big\downarrow \\ B \end{array}$$

Recall that there is a canonical scheme structure over each (closed) point of B, thus indeed this is as a collection of fibers, which are scheme, parametrized by closed points of B. To be a collection of our objects of interest, each fiber should *be* such an object. Thus we may impose that each fiber be isomorphic to a curve, or a surface, or even a map, or whatever it is that we are trying to classify. To make this work we often need to impose extra structures on the morphism. For example we may need to attach a map to each fiber so that we may have a family of maps.

This notion of a morphism being a collection of fibers parametrized by the base space should be reminiscent of the definition of a *fiber bundle*. Indeed, the reason one can think of a fiber bundle

$$\begin{array}{c} E \\ \downarrow \\ B \end{array}$$

as a collection of fibers parametrized by the points of the base space is precisely given by the condition of local triviality. That is, locally in B, the bundle looks like a trivial bundle:

$$\begin{array}{c} U \times F \\ \downarrow \\ U \end{array}$$

where U is the local neighborhood and F is the fiber. In other words, one can think of the fibers are "continuously varying" along the base.

In the world of schemes, the analogue to the notion of continuous varying fibers is flatness. Indeed, **we will only ever consider flat families**. This is a good condition to have and relates the geometry of the base to that of the fibers. Essentially, we only allow a base that "glues the fibers in a continuous fashion".

In addition, it is common that properness is imposed on the morphism. Properness for morphism of schemes can be roughly thought of as having compact fibers. Throughout this article, all of our families will be proper. This is more of a direct consequence of the types of objects we are considering, and not particularly an imposed condition.

We will denote the (flat) family

$$\begin{array}{c} X \\ \downarrow \\ B \end{array}$$

as X/B or simply (by abuse of notation) X if there is no risk of confusion. We will call the morphism itself the *structure morphism* of the family. Even though technically the morphism *is* the family, this language indicates that one should really think of the family as the collection of fibers, and the morphism as a tool to bind them together along a parameter space.

Some examples:

Example 1.1 If our goal is to classify all curves up to isomorphism, i.e. $S = \{\text{curves}\}$ and $\sim$ is isomophism of schemes, then a family of curves is a morphism $X \to B$ such that the fiber over every geometric point of B is isomorphic to a curve.

Example 1.2 If say we are interested in classifying maps from curves to some fixed space $\mathbb{P}^r$, we may define a family of such maps to be

$$\begin{array}{ccc} X & \xrightarrow{\mu} & \mathbb{P}^r \\ {\scriptstyle\pi}\big\downarrow & & \\ B & & \end{array}$$

such that π is a family of curves. Thus μ restricted to each fiber is a map from a curve to $\mathbb{P}^r$. In this particular case the structure morphism for the family of maps of curves is π, and we consider the map μ to be an example of extra structure we impose on the family.

We will also need the notion of *pullbacks of families*. This allows us to "change the parametrization" of the family (in a way compatible with operations in the category of schemes). More precisely, if X/B is a family parametrized by B, and $\varphi : B' \to B$ is a morphism of schemes, then we want an induced family over B', denoted by $\varphi^* X/B'$, called the *pullback family along* φ. This pullback family is defined as the fiber product $B' \times_B X$.

This definition also explains the term "base change". When we pull a family back along a morphism, we are literally "changing the base" of this family to another one.

One might again notice, this is analogous to pulling back a bundle along a morphism into the base.

There should also be a notion of equivalence of families over a common base compatible with the pullback operation. That is, we should have an equivalence relation on the set of families parametrized by a common base scheme B. More precisely, we have an equivalence relation on the set of families over B, such that if X/B and X'/B are equivalent families, then for any morphism $\varphi : B' \to B$, we have $\varphi^* X/B' \sim \varphi^* X'/B'$ (as families over B'). Intuitively, we should formulate some way of saying that families over the same base contain the same data are considered the same; and this notion should be compatible with any possible re-parametrization. Naturally, the notion of equivalency of families is most often given by an isomorphism on the total space making the obvious diagram commute. Of course, if the families we are considering has no extra structures, then this is literally just saying the maps are isomorphic. However, when we *do* have extra structures on the structure morphisms, we will always have to impose the condition that the structures be respected. The same is also true for the pullback operation. The precise notion of what it means to respect these structures depend heavily on the types of objects and types of structure, and they are defined on an ad-hoc basis. It is indeed often a non-trivial endeavor to come up with the right definitions for these notions when tackling a classification problem; and one can only get a good sense of what to look for with experience.

Example 1.3 For example, the first moduli problem we will investigate considers families of 4-tuples of points in $\mathbb{P}^1$, then in fact a family of such objects parametrized by B is the projection morphism $\pi : B \times \mathbb{P}^1 \to B$ together with four sections: morphisms $\sigma_i : B \to B \times \mathbb{P}^1$ that are disjoint. Thus in this case the structure morphism is π while the sections are considered extra structures. Now if there are two families over a common base B:

$$\begin{array}{c} B \times \mathbb{P}^1 \\ \pi \big\downarrow \big\uparrow \sigma_i \\ B \end{array} \quad \text{and} \quad \begin{array}{c} B \times \mathbb{P}^1 \\ \pi' \big\downarrow \big\uparrow \sigma_i' \\ B \end{array}$$

then to say that these two bundles have the same data is to say two things:

1. The fibers are all the same, i.e. the maps are isomorphic, so there is an isomorphism
$$\varphi\colon B \times \mathbb{P}^1 \to B \times \mathbb{P}^1$$
making the following diagram commute:
$$\begin{array}{ccc} B \times \mathbb{P}^1 & \xrightarrow{\varphi} & B \times \mathbb{P}^1 \\ & {\scriptstyle \pi} \searrow \quad \swarrow {\scriptstyle \pi'} & \\ & B & \end{array}$$

2. The sections should all be compatible under this isomoprhism, so we must have the following diagram commute for any $1 \leq i \leq 4$:
$$\begin{array}{ccc} B \times \mathbb{P}^1 & \xrightarrow{\varphi} & B \times \mathbb{P}^1 \\ & {\scriptstyle \sigma_i} \nwarrow \quad \nearrow {\scriptstyle \sigma_i'} & \\ & B & \end{array}$$

1.3 The Moduli Functor

The above notions and conditions of families, pullbacks, and equivalence of families can all be said succinctly by saying we have a contravariant functor:

$$\begin{aligned} F\colon \mathbf{Sch} &\to \mathbf{Set} \\ B &\mapsto \{\text{equivalence classes of families over } B\} \end{aligned}$$

This functor sends a morphism $\varphi : B' \to B$ to a morphism of sets

$$\begin{aligned} F(B) &\to F(B') \\ [X/B] &\mapsto [\varphi^* X/B'] \end{aligned}$$

We will term the functor associated to our choice of objects, families, and equivalence relations a *moduli problem*.

1.4 The Fine Moduli Space

Now that we have properly formulated the setting for a moduli problem, we will now introduce how an "answer" to a moduli problem, in the form of a scheme meeting some conditions, might be considered "good enough" to be called a moduli space for the moduli problem. The first set of criteria we will introduce is that of the *fine moduli space*, and represents the most strict requirements for a scheme to be considered a satisfactory moduli space, and in many respects whose existence is the best-case scenario.

The idea is that the fine moduli space should be a base scheme parametrizing a family of objects such that it is "universal among all families".

Definition 1.4 A *universal family* for a moduli problem F is a family U/M such that for any family X/B there exists a unique morphism $\kappa : B \to M$ such that $\kappa^* U$ is equivalent to X as families over B. We call the base M a *fine moduli space* for the moduli problem F.

There is an equivalent, categorical formulation of this definition. In terms of *representable functors*.

Definition 1.5 (Functor of Points) Let $\mathbf{C}$ be a locally small category (e.g. $\mathbf{Sch}$), and $Y \in \mathbf{C}$ an object. Define the *functor of points* as the contravariant functor

$$\begin{aligned} h_Y \colon \mathbf{C} &\to \mathbf{Set} \\ B &\mapsto \mathrm{Hom}_{\mathbf{C}}(B, Y) \\ [\varphi : B' \to B] &\mapsto [\beta \overset{\varphi^\#}{\longmapsto} \beta \circ \varphi] \end{aligned}$$

where $\beta \in h_Y(B) = \mathrm{Hom}_{\mathbf{C}}(B, Y)$.

Theorem 1.6 (Yoneda's Lemma) *Let $F : \mathbf{C} \to \mathbf{Set}$ be a contravariant functor from a locally small category $\mathbf{C}$ to the category of sets. Then for each object Y of $\mathbf{C}$, the natural transformations from h_Y to F are in bijection with the elements of $F(Y)$, i.e. there is a bijection*

$$\mathrm{Nat}(h_Y, F) \cong F(Y)$$

which sends each natural transformation $\Psi : h_Y \to F$ to $u = \Psi(Y)(\mathrm{id}_Y)$; and given an object u in $F(Y)$, the corresponding natural transformation is defined by $\Psi(f) = F(f)(u)$ for $f \in h_Y$.

Definition 1.7 (Representable Functor) Let F be a functor $F : \mathbf{C} \to \mathbf{Set}$, where $\mathbf{C}$ is a locally small category, F is called *representable* if F is isomorphic to h_Y for some object Y in $\mathbf{C}$.

By Yoneda's lemma, in order to get a representation for the functor F, we want to know when the natural transformation induced by an object u in $F(Y)$ is an isomorphism. So we introduce the following terminology:

Definition 1.8 (Universal Element) A *universal element* for the functor $F : \mathbf{C} \to \mathbf{Set}$ is a pair (A, μ) consisting of an object A of $\mathbf{C}$ and an element $\mu \in F(A)$ such that for every pair (X, v) with X and object of $\mathbf{C}$ and $v \in F(X)$ there exists a unique morphism (element in $h_A(X) = \mathrm{Hom}(X, A)$) $f : X \to A$ such that $F(f)(\mu) = v$

The condition is equivalent to the assertion that μ induces an isomrphism $\Psi : h_A \to F$ under the identification given by Yoneda's lemma. Therefore a functor F being representable by an object Y via an isomorphism $\Psi : h_Y \to F$ is equivalent to the existence of the universal element (A, μ). In this case we can say the universal element (A, Ψ) *represents* F.

Proposition 1.9 *A family U/M (with structure morphism μ) is a universal family for the moduli problem F if and only if the pair (M, Ψ) represents F, where we identify μ and Ψ via Yoneda's lemma.*

Proof Suppose that the moduli problem F is representable by a pair (M, Ψ), where M is a scheme and $\Psi : h_M \to F$ is an isomorphism of functors. Since Ψ is an isomorphism, for any scheme B and any element $[X \xrightarrow{\alpha} B] \in F(B)$, there exists a unique $\varphi \in h_M(B) = \mathrm{Hom}(B, M)$ such that $\Psi(B)(\varphi)(\alpha) = \varphi$. We claim that the family $\lambda = \Phi(M)(\mathrm{id}_M) : U \to M$ is universal. To show that we must show that any arbitrary family X/B (with structure morphism α) is a pullback of U/M along a unique morphism. We claim that this unique morphism is φ. To see that, we consider the commutative diagram:

$$\begin{array}{ccc}
\mathrm{id}_M & \longmapsto & \varphi^{\#}(\mathrm{id}_M) = \mathrm{id}_M \circ \varphi = \varphi \\
\downarrow & \begin{array}{ccc} h_M(M) & \xrightarrow{\varphi^{\#}} & h_M(B) \\ {\scriptstyle \Psi(M)}\downarrow & & \downarrow{\scriptstyle \Psi(B)} \\ F(M) & \xrightarrow{F(\varphi)} & F(B) \end{array} & \downarrow \\
\Psi(M)(\mathrm{id}_M) = \lambda & \longmapsto & F(\varphi)(\lambda) = \Psi(B)(\varphi) = \alpha
\end{array}$$

Now by definition of the pullback family along φ, $F(\varphi)(\lambda)$ is the family $U \times_M B \to B$. Thus the equality on the lower right-hand corner asserts that the arbitrary family X/B (with structure morphism α) is equivalent to $U \times_M B \to B$ as families over B.

Conversely, suppose U/M (with structure morphism μ) is a universal family. The definition of universal family implies that for any pair (B, v) where B is a scheme, and $v : Z \to B$ a family over B, then there exists a unique morphism $\varphi : B \to M$ such that pulling back the universal family along φ gives a family that is equivalent to Z/B. But this is equivalent to the assertion that (M, μ) is a universal element for F. Thus (A, Ψ) represents F. □

The next result is that the fine moduli space is unique up to unique isomorphism. As a result, the universal family for a given moduli problem, if it exists, is unique up to equivalence of families.

Proposition 1.10 *A universal family U/M for a moduli functor F, if it exists, is unique up to equivalence of families over M. Therefore, a fine moduli space M is unique.*

Proof This is a direct consequence of the fact that the universal family is defined via a universal property. □

Thus we can speak of *the* fine moduli space for a moduli problem, if it exists.

Points of the fine moduli space Recall that our goal was to find the most suitable scheme structure on the set $S/\sim_S$. Indeed, geometric points of the fine moduli space is in bijection with the isomorphism classes of objects: by the definition of universal family, there is a bijection between the set of equivalence classes of families over an arbitrary scheme B and the set of morphisms $B \to M$. Now if we take $B = \bullet = \mathrm{Spec}(\mathbb{C})$ (in fact we can take $\mathrm{Spec}(k)$ for any k a field, we really just need a one-point scheme), then a family over $\bullet$, i.e. a morphism $S \to \bullet$ is just S and the equivalence of families coincides with equivalence of objects; on the other hand morphisms $\bullet \to M$ are the geometric points of M. In other words the $\mathbb{C}$-valued points of M are in one-to-one correspondence with isomorphism classes of objects. In particular if M happens to be something nice such as a variety, then the closed points are in one-to-one correspondence with the isomorphism classes, which is what we intuitively would want to have.

1.5 The Coarse Moduli Space

It is in fact the case that the conditions for the existence of a fine moduli space is too stringent in many cases.

The simplest way to immediately relax the conditions is to not insist on an isomorphism of functors but rather just a natural transformation $\Psi : h_Y \to F$.

Definition 1.11 (Coarse moduli space) A *coarse moduli space* for a moduli functor F is a pair (M, v) where M is a scheme and $v : F \to h_M$ is a natural transformation such that

1. (M, v) is initial among all such pairs (defined precisely below).
2. The set map $v(\bullet) : F(\bullet) \to \mathrm{Hom}(\bullet, M)$ is a bijection.

That (M, v) is initial means that given any other pair (M', v') where $v' : F \to h_{M'}$ is a natural transformation, there exists a unique morphism of schemes $\psi : M \to M'$ such that, if we denote by $\Psi : h_M \to h_{M'}$ the natural transformation induced by ψ under Yoneda's lemma, we have $v' = \Psi \circ v$. In other words, every natural transformation $F \to h_{M'}$ factors uniquely through v.

It is in by last fact that we are able to say that h_M is the representable functor "closest" to F.

As one may expect, a fine moduli space is a coarse moduli space, if it exists. This is a simple fact, tave $v = \Psi^{-1}$ where Ψ is the isomorphism of functors associated to the fine moduli space.

Another unsurprising fact is that a coarse moduli space is unique up to unique isomorphism, and in fact only Condition 1. in the definition is needed to show this:

Proposition 1.12 *If a coarse moduli space for a moduli functor F exists, then it is unique up to unique isomorphism.*

Proof As before, condition 1. is a statement that (M, v) satisfies an initial property, i.e. an universal property. □

Remark on this definition The above definition of the coarse moduli space is not what one would find in the literature. The term coarse moduli space is a much more general object that is defined for together with a *stack*. Stacks are very roughly, a class of objects that are an enlargement of the category of schemes. In fact, a stack is a functor, with some other stuff. We won't go into more detail to try to explain any of this, all that matters is that one should be mindful of the use of this term in the literature, that the coarse moduli space can be a much more general object in contexts that are more general than ours.

Chapter 2

The Moduli Space of Pointed Curves

This chapter concerns the construction of the fine moduli space to the moduli problem of classifying projective rational curves with n marked points up to isomorphism that respects the marked points.

To begin with, we first look at the special case of projective smooth rational curves, which we will prove are all isomorphic to $\mathbb{P}^1$. Thus isomorphism classes of projective smooth rational curves with n marked points are equivalent to isomorphism classes of automorphisms of $\mathbb{P}^1$ that respects an extinguished n-tuple of marked points. A construction of the fine moduli space classifying marked projective smooth rational curves, denoted $M_{0,n}$ will be given. We will then give the statement (without proof) from Knudsen [10] of the existence of a fine moduli space classifying n-pointed projective rational curves with a certain stability condition imposed, again up to isomorphism respecting the marked points. This space, denoted $\overline{M}_{0,n}$ will contain $M_{0,n}$ as a dense open subset, and thus is an instance of a *compactification* of a moduli space.

Throughout this article, we use the following definition of $\mathbb{P}^r$:

Definition 2.1 The *complex projective r-space* is the set of all lines through the origin in $\mathbb{C}^{r+1}$. Equivalently, it is all the points in $\mathbb{C}^{r+1}$ where we identify points which are non-zero scalar multiples of each other. Thus we can represent a point in $\mathbb{P}^r$ by an $r+1$ tuple called its *homogeneous coordinate*. Of course, the homogenous coordinate of a point is not unique, any non-zero scalar multiple of one is also a homogenous coordinate of the same point.

The complex projective line $\mathbb{P}^1$ is the simplest example of a projective curve.

2.1 Classifying n-tuples up to Projective Equivalence

Definition 2.2 Consider a pair of ordered n-tuples

$$\boldsymbol{p} = (p_1, \dots, p_n)$$

$$\boldsymbol{q} = (q_1, \dots, q_n)$$

in $\mathbb{P}^1$ with distinct entries, i.e. $p_i \neq p_j$ and $q_i \neq q_j$ for any $i \neq j$. We say that $\boldsymbol{p}$ and $\boldsymbol{q}$ are *projectively equivalent* if there exists an automorphism $\varphi : \mathbb{P}^1 \to \mathbb{P}^1$ such that $\varphi(p_i) = q_i$ for every $i = 1, \dots, n$.

Unless otherwise noted, an *n-tuple* will be an n-tuple in $\mathbb{P}^1$ with distinct entries.

Our goal is to construct a fine moduli space for the moduli problem of classifying n-tuples up to projective equivalence. In order to do so we must specify our notion of families of n-tuples, and a notion of equivalence of families.

At the moment we will restrict our attention to the particular case where $n = 4$. The reason is that one can move up to three points under an automoprphism of $\mathbb{P}^1$ to any desired points. As such, for 3-tuples there is only one equivalence class. Naturally, the case for $n = 4$ is the first non-trivial case, and we shall see that once we have constructed the fine moduli space for this case, it is not difficult to generalize to arbitrary n.

We will give a special name for a 4-tuple, for no other reason but convenience.

Definition 2.3 A *quadruple* of points in $\mathbb{P}^1$ is an ordered tuple of four distinct points $\mathbf{p} = (p_1, p_2, p_3, p_4)$ in $\mathbb{P}^1$.

Proposition 2.4 *The set of all quadruples*

$$Q = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \setminus \textit{diagonals}$$

form an algebraic variety, thus in particular it has a scheme structure.

By *diagonals* we mean any pair of entries with the same value, which is not allowed by our definition of quadruples.

Proof The space $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ has the structure of a quasi-projective variety by the Seagre embedding. The diagonals are closed subsets, thus Q is a quasi-projective variety. □

Definition 2.5 Two quadruples $\mathbf{p}$ and $\mathbf{p}'$ are called *projective equivalent* if there exists an automorphism $\varphi : \mathbb{P}^1 \xrightarrow{\sim} \mathbb{P}^1$ of the projective line such that $\varphi(p_i) = p'_i$ for every $i = 1, 2, 3, 4$.

We are interested in classifying quadruples of points in $\mathbb{P}^1$ up to projective equivalence. In particular we would like to seek a scheme that can be naturally identified with the set of equivalence classes. In other words, we would like to pose a moduli problem $F : \mathbf{Sch}^{\mathrm{opp}} \to \mathbf{Set}$. We will need to define the proper notion of families, and a proper equivalence relation on families.

Definition 2.6 (family of quadruples) A *family of quadruples* (over a base scheme B) is the following diagram/collection of morphisms:

$$\begin{array}{c} B \times \mathbb{P}^1 \\ \pi \downarrow \quad \uparrow\uparrow\uparrow\uparrow \sigma_i \\ B \end{array}$$

where $\pi : B \times \mathbb{P}^1 \to B$ is the projection, and the four sections

$$\sigma_i : B \to B \times \mathbb{P}^1, \quad i = 1, 2, 3, 4$$

are disjoint, i.e. over each geometric point $b \in B$,

$$\sigma_j(b) \neq \sigma_k(b) \text{ if } j \neq k$$

We can view this diagram as such: Over each point $b \in B$, the fiber of π is a copy of the whole projective plane $\mathbb{P}^1$, while the sections σ_i single out four distinct points. Therefore, the fiver over each point in B corresponds to a quadruple.

Family of quadruples as a morphism In our case the structure morphism is π while the sections are "extra structures".

Now we specify our notion of equivalence of families.

Definition 2.7 Two families of quadruples over a fixed base scheme B:

$(B, \sigma_1, \sigma_2, \sigma_3, \sigma_4)$ and $(B, \sigma'_1, \sigma'_2, \sigma'_3, \sigma'_4)$ are *equivalent* if there is an automorphism of schemes $\varphi : B \times \mathbb{P}^1 \to B \times \mathbb{P}^1$ making the following diagram commute for $i = 1, 2, 3, 4$:

$$\begin{array}{ccc} B \times \mathbb{P}^1 & \xrightarrow{\varphi} & B \times \mathbb{P}^1 \\ \pi \downarrow \uparrow \sigma_i & & \pi' \downarrow \uparrow \sigma'_i \\ B & = & B \end{array}$$

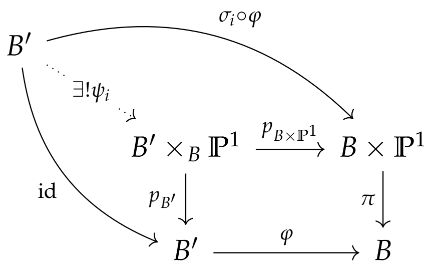
Now that we have defined families of objects, and a compatible notion of equivalence of families, we need the notion of pullbacks. Pullbacks of families will be given by fiber products, but care must be taken in specifying what happens to the sections when we form the fiber product.

Definition 2.8 (pullback family of quadruples) Let $B \times \mathbb{P}^1 \xrightarrow{\pi} B$ (with its sections σ_i) be a family of quadruples. Let $\varphi : B' \to B$ be a morphism of schemes. Then we have the following diagram:

$$\begin{array}{ccc} B' \times_B \mathbb{P}^1 = B' \times_B B \times \mathbb{P}^1 & \xrightarrow{p_{B\times\mathbb{P}^1}} & B \times \mathbb{P}^1 \\ {\scriptstyle p_{B'}}\downarrow\uparrow{\scriptstyle \sigma_i\circ\varphi} & & {\scriptstyle \pi}\downarrow\uparrow{\scriptstyle \sigma_i} \\ B' & \xrightarrow{\varphi} & B \end{array}$$

where $p_{B\times\mathbb{P}^1}$ and $p_{B'}$ are the unique maps associated to the fiber product.

Now consider the map $\sigma_i \circ \varphi$, and using the universal property of the fiber product, we have the following commutative diagram:



in this way we can define sections $\psi_i : B' \times B' \times_B \mathbb{P}^1$.

Then the *pullback family* of $B \times \mathbb{P}^1 \xrightarrow{\pi} B$ along φ will then be the following portion of the above diagram:

$$\begin{array}{c} B' \times \mathbb{P}^1 \\ {\scriptstyle p_{B'}}\downarrow\uparrow{\scriptstyle \psi} \\ B' \end{array}$$

where we identified $B' \times_B B \times \mathbb{P}^1 \cong B' \times \mathbb{P}^1$. Thus the pullback family is a family $B' \times \mathbb{P}^1 \xrightarrow{p_{B'}} B'$ with sections ψ, i.e. $p_{B'} \circ \psi_i = \mathrm{id}_{B'}$ for every $i = 1, 2, 3, 4$. The sections are disjoint by the disjointness of the sections σ_i in the original family.

By abuse of notation, we will denote ψ as simply $\sigma \circ \varphi$. Thus we may rewrite the pullback family we have just defined to be

$$\begin{array}{c} B' \times \mathbb{P}^1 \\ {\scriptstyle p_{B'}}\downarrow\uparrow{\scriptstyle \sigma_i\circ\varphi} \\ B' \end{array}$$

We will do this for families of other objects in later parts of this paper and their pullbacks whenever there are sections involved.

Now we have all the necessary ingredients to define the moduli problem of classifying quadruples in $\mathbb{P}^1$ up to projective equivalence.

The following proposition will be important in describing the isomoprhism classes of quadruples.

Proposition 2.9 *Given any triple of distinct points $p_1, p_2, p_3 \in \mathbb{P}^1$, there exists a unique automorphism $\varphi : \mathbb{P}^1 \to \mathbb{P}^1$ such that*

$$p_1 \mapsto 0, \quad p_2 \mapsto 1, \quad p_3 \mapsto \infty$$

Proof We first consider the case that all three points are finite. Then we can write them in terms of homogeneous coordinates as

$$p_1 = \begin{bmatrix} x_1 \\ 1 \end{bmatrix}, \quad p_2 = \begin{bmatrix} x_2 \\ 1 \end{bmatrix}, \quad p_3 = \begin{bmatrix} x_3 \\ 1 \end{bmatrix}.$$

Let

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{PGL}(2)$$

Then p_1 is sent to $0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ if and only if $b = -ax_1$. Also, p_3 is sent to $\infty = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ if and only if $d = -cx_3$. Lastly, p_2 is sent to $1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ if and only if

$$\frac{ax_2 - ax_1}{cx_2 - cx_3} = 1.$$

Since we assume that the points are distinct, $x_2 \neq x_3$ thus we can re-write this as

$$a = c\frac{x_2 - x_3}{x_2 - x_1}$$

Now we notice that $a, c \neq 0$. If they are both zero then the matrix will have determinant zero, thus not it will not represent an automoprhism. On the other hand, if $a = 0$ then $b = 0$; if $c = 0$ then $d = 0$, both cases will also force the matrix to have zero determinant. As such, we can assume $c = 1$ by re-scaling, to obtain

$$a = \frac{x_2 - x_3}{x_2 - x_2}$$

By the above argument we conlcude that an automorphism $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ gives the desired result if and only if

$$a = \frac{x_2 - x_3}{x_2 - x_2}, \quad b = -ax_1, \quad c = 1, \quad d = -cx_3$$

(up to scaling by a constant). Such a matrix obviously exists, thus is concludes the proof for the case that all three points are finite.

Now we suppose that one of the three points is the point at infinity. Then there are three cases: (As before, let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PGL}(2)$)

1. $p_1 = \infty$.

   The matrix sends $p_1 = \infty = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}$ to 0 if and only if $a = 0$. The matrix sends $p_3 = 1 = \begin{bmatrix} x_3 \\ 0 \end{bmatrix}$ to ∞ if and only if $d = -cx_3$. Lastly, the matrix sends $p_2 = \begin{bmatrix} x_2 \\ 1 \end{bmatrix}$ to 1 if and only if

   $$b = c(x_2 - x_3)$$

   But as before, it suffices to assume $c = 1$ to obtain $b = x_2 - x_3$. Therefore the unique automorphism is

   $$\begin{pmatrix} 0 & x_2 - x_3 \\ 1 & -x_3 \end{pmatrix}$$

2. $p_2 = \infty$.

   The matrix sends p_1 to $0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ if and only if $b = -ax_1$. Also, the matrix sends p_3 to $\infty = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ if and only if $d = -cx_3$. Lastly, the matrix sends $p_2 = \infty = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ to $1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ if and only if $ax_2 = cx_2$, i.e. $a = c$. And since we cannot have both a and c be zero, it suffices to assume $a = c = 1$. Therefore the unique automorphism is

   $$\begin{pmatrix} 1 & -x_1 \\ 1 & -x_3 \end{pmatrix}$$

3. $p_3 = \infty$.

   The matrix sends $p_3 = \infty = \begin{bmatrix} x_3 \\ 0 \end{bmatrix}$ to itself if and only if $c = 0$. Also, the matrix sends p_1 to $0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ if and only if $b = -ax_1$. Lastly, the matrix sends p_2 to $1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ if and only if

   $$a(x_2 - x_1) = d$$

but a cannot be zero, otherwise b will be zero which will force the matrix to have determinant zero. Therefore it suffices to assume $a = 1$ to obtain $d = x_2 - x_1$. Therefore the unique automorphism is

$$\begin{pmatrix} 1 & -x_1 \\ 0 & x_2 - x_1 \end{pmatrix}$$ □

With this result in mind, we can give the following definition.

Definition 2.10 Let $\boldsymbol{p} = (p_1, p_2, p_3, p_4)$ be a quadruple, and let φ_p denote the unique automorphism on $\mathbb{P}^1$ that sends $p_1 \mapsto 0, p_2 \mapsto 1, p_3 \mapsto \infty$. Then the value $\lambda(\boldsymbol{p}) := \varphi_p(p_4) \in \mathbb{P}^1$ is called the *cross-ratio* of $\boldsymbol{p}$.

It is clear from the definition of automorphism that $\lambda(\boldsymbol{p}) \neq 0, 1, \infty$ for any quadruple $\boldsymbol{p}$.

Corollary 2.11 *Every quadruple $\boldsymbol{p}$ is projectively equivalent to $(0, 1, \infty, \lambda(\boldsymbol{p}))$. Therefore two quadruples are projectively equivalent if and only if they have the same cross ratio.*

Proof Follows immediately from Proposition 2.9 and Definition 2.10 □

As a consequence, the fine moduli space classifying all quadruples up to projective equivalence, if it exists, its closed points must be in a natural bijection with $\mathbb{P}^1 \setminus \{0, 1, \infty\}$.

2.1.1 Existence of the fine moduli space classifying quadruples

Theorem 2.12 *The variety*

$$M_{0,4} := \mathbb{P}^1 \setminus \{0, 1, \infty\}$$

is a fine moduli space for the classification of quadruples up to projective equivalence.

Proof We will prove the result by constructing a universal family over $M_{0,4}$. The family we will construct is the so-called *tautological family* over $M_{0,4}$, which is a family with the property that the fiber over any $q \in M_{0,4}$ is a quadruple (specified by the values of the sections) with cross ratio q. The (claimed) universal family is the diagram

$$\begin{array}{c} M_{0,4} \times \mathbb{P}^1 \\ {\scriptstyle p_{M_{0,4}}}\downarrow \quad \uparrow\uparrow\uparrow\uparrow {\scriptstyle \tau_i} \\ M_{0,4} \end{array}$$

where the first three sections are constant maps with values $0, 1$, and ∞, and the fourth section sends a point to itself. Let

$$\begin{array}{c} B \times \mathbb{P}^1 \\ {\scriptstyle p_B}\big\downarrow \quad \big\uparrow\big\uparrow\big\uparrow\big\uparrow{\scriptstyle \sigma_i} \\ B \end{array}$$

be an arbitrary family of quadruples. Define the morphism $\varphi : B \to M_{0,4}$ by the following composition of morphisms:

$$B \to \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \setminus \text{diagonals} \to \mathrm{Aut}(\mathbb{P}^1) \times \mathbb{P}^1 \to M_{0,4} = \mathbb{P}^1 \setminus \{0, 1, \infty\}$$

$$b \mapsto (\sigma_1(b), \sigma_2(b), \sigma_3(b), \sigma_4(b)) \mapsto (\alpha, \sigma_4(b)) \mapsto \alpha(\sigma_4(b)) = \lambda(\sigma_4(b))$$

where α is the unique automorphism that sends $(\sigma_1(b), \sigma_2(b), \sigma_3(b))$ to $(0, 1, \infty)$. Then the pullback of the tautological family along φ is by definition the following family over B:

$$\begin{array}{c} B \times \mathbb{P}^1 \\ {\scriptstyle p_B}\big\downarrow \quad \big\uparrow\big\uparrow\big\uparrow\big\uparrow{\scriptstyle \zeta_i} \\ B \end{array}$$

where ζ_i is the map given by the universal property of the fiber product from B to $B \times \mathbb{P}^1$:

$$\begin{array}{ccccc} B & & \xrightarrow{\quad \tau_i \circ \varphi \quad} & & \\ & \overset{\exists ! \zeta_i}{\searrow} & & & \\ & & B \times_B \mathbb{P}^1 & \xrightarrow{p_{M_{0,4} \times \mathbb{P}^1}} & M_{0,4} \times \mathbb{P}^1 \\ & {\scriptstyle \mathrm{id}_B} & {\scriptstyle p_B}\big\downarrow & & {\scriptstyle p_{M_{0,4}}}\big\downarrow\big\uparrow{\scriptstyle \tau_i} \\ & & B & \xrightarrow{\quad \varphi \quad} & M_{0,4} \end{array}$$

From this diagram we can also get a description of what the sections ζ_i are. First, from the lower-right square (i.e. the fiber diagram) we must have that

$$p_{M_{0,1} \times \mathbb{P}^1} = (\varphi, \mathrm{id}_{\mathbb{P}^1})$$

thus in order to have $\zeta_i \circ p_{M_{0,1} \times \mathbb{P}^1} = \tau_i \circ \varphi$ we must have

$$\zeta_i(b) = (b, \tau_i(\varphi(b))) \tag{2.1}$$

(by $\tau_i(\varphi(b))$ we mean the second component of the image of $\varphi(b)$).

We claim that the pullback family is equivalent to the family (B, σ_i). To show this, we need to show the existence of an automorphism of $B \times \mathbb{P}^1$ that respects the two families and their sections.

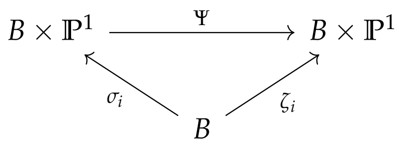
We define the automorphim

$$\begin{aligned}\Psi\colon B \times \mathbb{P}^1 &\to B \times \mathbb{P}^1\\ (b, \boldsymbol{p}) &\mapsto (b, \psi_b(\boldsymbol{p}))\end{aligned}$$

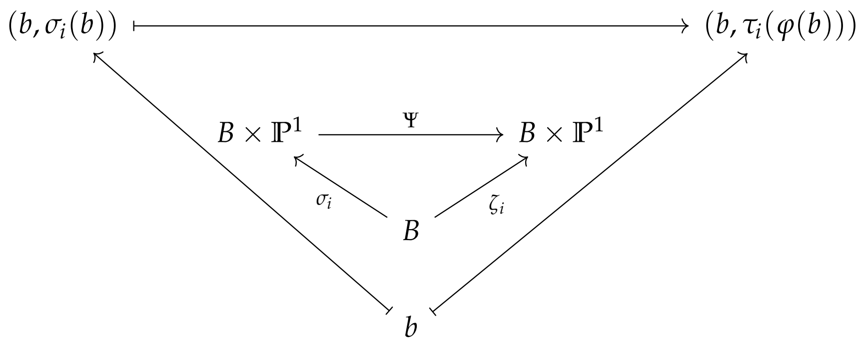
where ψ_b is a morphism on $\mathbb{P}^1$ defined by

$$\psi_b(p) = \begin{cases} 0 & p = 0 \\ 1 & p = 1 \\ \infty & p = \infty \\ \tau_4(\lambda(p_4)) & \text{otherwise} \end{cases}$$

Since the projection maps on the two families are the same, we only need to check that the sections are respected by the automorphism. That is, we need to show that following diagram commutes for all $i = 1, 2, 3, 4$:



Using the description of ζ_i (2.1), we have the explicit mappings



Then it is straigtforward to check using the definition of φ that this diagram commutes for every i.

It remains to check that the morphism φ is unique among all such morphisms with the pullback property. However, this follows from the way we defined b: the image of a point b is the cross ratio of the fiber over b. This clearly determines κ completely if it exists.

This shows that $M_{0,4}$ which is the base of the universal family is a fine moduli space for the moduli problem of classifying quadruples up to projective equivalence. □

2.1.2 Generalizing to n-tuples

We will now generalize the result to obtain the fine moduli space for the classification of n-tuples up to projective equivalence, where $n \geq 4$. The definitions of family of quadruples and their equivalence are easily extended to the case for n-tuples.

First, the notions of families of n-tuples, their equivalence, all related notions of the moduli problem is the same for the case of quadruples. Be aware that each moduli problem considers a fixed $n \geq 4$.

Theorem 2.13 *For $n \geq 3$, the fine moduli space for the moduli probelm of classifying n-tuples in $\mathbb{P}^1$ up to projective equivlance is*

$$M_{0,n} \cong \underbrace{M_{0,4} \times \cdots \times M_{0,4}}_{(n-3)\ \text{factors}} \setminus \bigcup \text{diagonals}$$

Proof The proof is almost exactly the same for the case of quadruples.

We claim the the following family defines a universal family:

$$\begin{array}{c} M_{0,n} \times \mathbb{P}^1 \\ {\scriptstyle p_{M_{0,n}}}\big\downarrow \big\uparrow {\scriptstyle \tau_i} \\ M_{0,n} \end{array}$$

where the sections τ_i, $4 \geq i \geq n$ are the projections from the $(n-3)$-fold product to its factors $M_{0,4} \subset \mathbb{P}^1$.

Let

$$\begin{array}{c} B \times \mathbb{P}^1 \\ {\scriptstyle p_B}\big\downarrow \big\uparrow {\scriptstyle \sigma_i} \\ B \end{array}$$

be an arbitrary family of n-tuples in $\mathbb{P}^1$ over the base scheme B. Define the unique morphism

$$\varphi : B \to M_{0,n}$$

which is defined by the following composition of maps

$$B \to \underbrace{\mathbb{P}^1 \times \cdots \times \mathbb{P}^1}_{n\ \text{factors}} \setminus \bigcup \text{diagonals} \to \text{Aut}(\mathbb{P}^1) \times \underbrace{\mathbb{P}^1 \times \cdots \times \mathbb{P}^1}_{(n-3)\ \text{factors}} \to M_{0,n}$$

defined by

$$b \mapsto (\sigma_1(b), \ldots, \sigma_n(b)) \mapsto (\alpha, \sigma_4(b), \ldots, \sigma_n(b)) \mapsto (\alpha(\sigma_4(b)), \ldots, \alpha(\sigma_n(b)))$$

where α is again the unique isomorphism sending the first three section to $0, 1, \infty$ respectively.

The the pullback along φ of the (claimed) universal family is the following diagram:

$$\begin{array}{c} B \times \mathbb{P}^1 \\ \pi \downarrow \uparrow \zeta_i \\ B \end{array}$$

where ζ_i is as before obtained from the universal property of the fiber product.

We claim that this pullback family along ν is equivalent to the family $B \times \mathbb{P}^1 \xrightarrow{p_B} B$ (with its sections σ_i). Indeed, define the automorphim

$$\begin{aligned} \Psi \colon B \times \mathbb{P}^1 &\to B \times \mathbb{P}^1 \\ (b, \boldsymbol{p}) &\mapsto (b, \psi_b(\boldsymbol{p})) \end{aligned}$$

where

$$\psi_b(\boldsymbol{p}) = \begin{cases} 0 & p = 0 \\ 1 & p = 1 \\ \infty & p = \infty \\ \tau_4(\lambda(p_4)) & \text{otherwise} \end{cases}$$

It is then again straightforward to check that Ψ gives an isomorphism of families. □

2.2 Classifying Pointed Smooth Rational Curves up to Isomorphism

A projective smooth rational curve is isomorphic to $\mathbb{P}^1$. Thus a projective smooth rational curve with n marked points is nothing but a copy of $\mathbb{P}^1$ with n marked points. We will show that the moduli problem of classifying n-pointed projective smooth rational curves up to isomorphism that respects the marked points to be the same as classifying n-tuples up to projective equivalence. We will have a slightly different definition of families of such objects than the one we had for tuples, but we will show that every family we define here is equivalent (as families) to a family of the form we defined in the previous section, i.e. a morphism whose source is the product of the base with $\mathbb{P}^1$. Thus justifying what we mean by these moduli problems are the same.

Definition 2.14 An *n-pointed projective smooth rational curve* is a projective smooth rational curve C together with n distinct *marked points* $p_1, \dots, p_n \in C$.

We denote an n-pointed curve by

$$(C, p_1, \ldots, p_n)$$

Definition 2.15 An *isomorphism* of n-pointed projective smooth rational curves

$$\varphi : (C, p_1, \ldots, p_n) \xrightarrow{\sim} (C', p'_1, \ldots, p'_n)$$

is an isomorphism $\varphi : C \xrightarrow{\sim} C'$ that respects the marked points, i.e. $\varphi(p_i) = p'_i$, for $i = 1, \ldots, n$.

Thus we see that the definition resembles that of projective equivalence, especially if we consider the curve C as just a copy of $\mathbb{P}^1$ (by *copy* we mean they are isomorphic as varieties).

As all curves that we shall be considering are projective, we suppress the word *projective* when speaking of them. We will however keep the adjectives *smooth* and *rational* as they are not only crucial properties to keep in mind but also in anticipation that we will consider non-smooth and non-rational curves later.

Definition 2.16 A *family of n-pointed smooth rational curves* (over a base scheme B) is a flat, proper morphism of schemes $\pi : X \to B$ with n distinct sections $\sigma_i : B \to X$ such that each fiber $\pi^{-1}(b)$ is isomorphic to a smooth rational curve. By *sections* we mean maps σ_i such that $\pi \circ \sigma_i = \mathrm{id}_B$. That is, a diagram

$$\begin{array}{c} X \\ {\scriptstyle\pi}\big\downarrow\big\uparrow{\scriptstyle\sigma_i} \\ B \end{array}$$

We can see from this definition that the sections over a point $b \in B$ single out n distinct points in the curve $\pi^{-1}(b)$, making the curve into an n-pointed curve.

We also need a notion of equivalence of families:

Definition 2.17 Two families of n-pointed smooth rational curves over a common base scheme B:

$$\pi : X \to B \text{ with sections } \sigma_i, \text{ and } \pi' : X' \to B \text{ with sections } \sigma'_i$$

are said to be *equivalent* or *isomorphic* if there exists an isomorphism $\varphi : X \to X'$ such that the following diagram commutes for each $i = 1, \ldots, n$:

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & X' \\ {\scriptstyle\pi}\big\downarrow\big\uparrow{\scriptstyle\sigma_i} & & {\scriptstyle\pi'}\big\downarrow\big\uparrow{\scriptstyle\sigma'_i} \\ B & = & B \end{array}$$

We still need to define the pullback of a family along a morphism. This will unsurprisingly be given by the fiber product, and sections given by the universal property of the fiber product.

Definition 2.18 Let $\pi : X \to B$ be a family of n-pointed smooth rational curves with sections σ_i, and $\varphi : B' \to B$ be a morphism. Then the *pullback family along* φ is the family

$$\begin{array}{c} B' \times_B X \\ {\scriptstyle p_{B'}}\big\downarrow\big\uparrow{\scriptstyle \zeta_i} \\ B' \end{array}$$

where as before the sections ζ_i are the unique maps from B' to $B' \times_B X$ given by the universal property of the fiber product.

We will show that the moduli problem of classifying n-pointed smooth rational curves up to isomorphism is equivalent to the moduli problem of classifying n-tuples in $\mathbb{P}^1$ up to projective equivalence. Hence justifying our treatment of the latter moduli problem in the previous section. In particular the fine moduli spaces we constructed for n-tuples will also be the appropriate fine moduli spaces for classifying n-pointed smooth ratoinal curves.

As mentioned before, the key to this equivalence is that projective smooth rational curves are the same as $\mathbb{P}^1$. We first establish this fact:

Proposition 2.19 (Moraru [12] Corollary 1.3.5) *Any projective smooth rational curve C is isomorphic to $\mathbb{P}^1$.*

Proof It suffices to show that any rational map $\varphi : C \to \mathbb{P}^1$ is a morphism (this fact is actually also more generally true where the target space is $\mathbb{P}^n$ for any n). We use the fact that C can be embedded in $\mathbb{P}^n$ for some n.

By definition of a rational map, we can write the value of φ at a point $[x_1 : \cdots : x_{n+1}]$ where φ is defined, viewed as embedded in $\mathbb{P}^n$ as:

$$\varphi(x_1 : \cdots : x_{n+1}) = [F_1(x_1, \ldots, x_{n+1}) : F_2(x_1, \ldots, x_{n+1})]$$

where $F_1, F_2 \in K(C) = \mathbb{C}[x_1, \ldots, x_{n+1}]$ are homogeneous polynomials.

Now let $p = [y_1 : \cdots : y_{n+1}]$ be any arbitrary point on the curve (not necessarily in the domain of φ), we will show that F_1 and F_2 do not vanish at p, thus showing that φ is in fact a morphism. By assumption C is smooth. We know that a projective curve C is smooth at a point p if and only if the local ring $\mathcal{O}_{C,p}$ of p is a discrete valuation ring (DVR) (Moraru [12] Proposition 1.2.13). Therefore the functions $F_1, F_2 \in \mathcal{O}_{C,p} \subset K(C)$ are in a DVR. So we can use properties of DVR to write

$$F_i = t^{k_i} u_i$$

where $t \in \mathcal{O}_{C,p}$ is a fixed uniformizer, $k_i \in \mathbb{Z}$, and $u_i \in \mathcal{O}_{C,p}$ are units. After a possible change of coordinates, we may assume that $k_1 \leq k_2$. Then

$$\begin{aligned}\varphi(y_1 : \cdots : y_{n+1}) &= [F_1(y_1, \ldots, y_{n+1}) : F_2(y_1, \ldots, y_{n+1})] \\ &= [t^{k_1} u_1(y_1, \ldots, y_{n+1}) : t^{k_2} u_2(y_1, \ldots, y_{n+1})] \\ &= [u_1(y_1, \ldots, y_{n+1}) : t^{k_2 - k_1} u_2(y_1, \ldots, y_{n+1})]\end{aligned}$$

The first component is non-zero at p since u_1 is a unit in $\mathcal{O}_{C,p}$, and the second component is non-zero since u_2 is also a unit in $\mathcal{O}_{C,p}$ and $t_{k_1} \leq t_{k_2}$. This proves the desired result.

Therefore we can conclude that any birational equivalence between C and $\mathbb{P}^1$ will in fact be an isomorphism. □

The following key proposition establishes the fact that any family of n-pointed smooth rational curves $\pi : X \to B$ (with sections σ_i) is isomorphic as families to a family of the form $B \times \mathbb{P}^1 \to B$ (with apprpriate sections). Unfortunately, we are unable to provide the proof, which is beyond the scope of this paper. According to our main source material, it is a slight modification of the proof of Proposition III.2.2 of Hartshorne [7].

Proposition 2.20 *Let $\pi : X \to B$ be a family of n-pointed smooth rational curves (with sections σ_i). Then there is a unique isomorphism $\varphi : X \to B \times \mathbb{P}^1$ making the family isomorphic to a family of the form*

$$\begin{array}{c} B \times \mathbb{P}^1 \\ \pi \big\downarrow \quad \big\uparrow\big\uparrow\big\uparrow\big\uparrow \gamma_i \\ B \end{array}$$

where the first three sections are the constant maps

$$\gamma_1(b) = 0, \quad \gamma_2(b) = 1, \quad \gamma_3(b) = \infty, \quad \forall b \in B$$

called the trivial family. *The name should remind one of a trivial bundle.*

Thus we can immediately conclude

Corollary 2.21 *For $n \geq 3$, classifying n-tuples of distinct points in $\mathbb{P}^1$ up to projective equivalence is equivalent to classifying n-pointed smooth rational curves up to isomorphism.*

Theorem 2.22 (Existence and Description of $M_{0,n}$) *For $n \geq 3$, there is a fine moduli space, denoted by $M_{0,n}$ for the moduli problem of classifying n-pointed curves up to isomorphism, given by the following:*

1. *$M_{0,3}$ is a single point.*

2. $M_{0,4} \cong \mathbb{P}^1 \setminus \{0, 1, \infty\}$.
3. *For* $n > 4$,

$$M_{0,n} \cong \underbrace{M_{0,4} \times \cdots \times M_{0,4}}_{(n-3)\ \mathit{factors}} \setminus \bigcup \mathit{diagonals}$$

Proof Due to Corollary 2.21, much of the work has already been done. From Theorem 2.12 and Theorem 2.13 we get 2. and 3. respectively. As for $M_{0,3}$, we know that there is only one equivalence class of 3-pointed smooth rational curves, therefore the moduli space is a single point. It is then trivial to verify that any family of triples is the pullback of the trivial family over a single point. □

2.3 Compactifying $M_{0,n}$

In general the moduli spaces $M_{0,n}$ we have constructed are not compact. For the purpose of studying the intersection theory on these moduli spaces, there is a need to compactify them. This will be done by adding additional points to the moduli space, in such a way that our moduli space becomes a (larger) moduli space for classifying a class of objects which are slight generalizations of n-pointed smooth rational curves, in such a way that $M_{0,n}$ is contained as a dense subset. This larger class of objects will be the so-called *stable n-pointed rational curves* (which are not necessarily smooth).

We begin by dropping the smoothness condition we have been imposing.

Definition 2.23 An *n-pointed rational curve* is a projective rational curve C together with n distinct *marked points* $p_1, \ldots, p_n \in C$, denoted by

$$(C, p_1, \ldots, p_n)$$

The notions of isomorphism of curves is still the same, i.e. an isomorphism of n-pinted rational curves is an isomorphism of curves that respects the marked points.

Next we identify a special class of rational curves called a *tree of projective lines*:

Definition 2.24 A *tree of projective lines* or *genus-0 nodal curve* is a connected curve with the following properties:

1. Each irreducible component is isomorphic to $\mathbb{P}^1$.
2. The points of intersection of the irreducible components are ordinary double points.

3. There are no closed circuits. That is, of a node is removed, the curve becomes disconnected. Equivalently, if δ is the number of nodes, then there are $\delta + 1$ irreducible components.

   An irreducible component of a tree of projective lines will be called a *twig*.

When we say ordinary double point, we mean that locally at the point of intersection, the curve is complex analytically isomorphic to a neighborhood of the origin in the zero-locus defined by the equation $xy = 0$ in $\mathbb{C}^2$.

We will drop the prefix "genus-0" since we will not be considering higher genus curves.

Now we can define the central object of interest for this section, stable curves, which are trees of projective lines with conditions on the marked points and nodes:

Definition 2.25 Let $n \geq 3$. A *stable n-pointed rational nodal curve* is a nodal curve $(C, p_1, \ldots, p_n)$ with n distinct marked points that are smooth points of the curve C, such that every twig contains at least three special points. A *special point* is a marked point or a node, i.e. a point of intersection with another twig.

Thus a stable n-pointed rational nodal curve is in particular an n-pointed rational nodal curve, which is also obvious from the terminology.

Again, The notion of isomoprhism of curves still holds for stable n-pointed rational nodal curves: all is needed is the isomorphism respect the marked points.

We will drop the words "rational" and "nodal" from now on, since all the curves we will be considering are rational, and we will indicate specifically when we are only considering smooth curves.

An *automorphism* of an n-pointed curve (with or without being stable or smooth) $(C, p_1, \ldots, p_n)$ is an isomorphism $\psi : C \xrightarrow{\sim} C$ that fixes each marked point. We will call a stable n-pointed rational curve *automorphism-free* if the identity is the only automorphism.

The term *stable* refers to the characteristic of being automorphism-free. The following proposition will justify this terminology by showing that stability as we have defined using special points is equivalent to the condition that the curve is automorphism-free.

Proposition 2.26 *An n-pointed curve is automorphism free if and only if it is a stable n-pointed curve.*

Proof Let φ be an automorphism of a stable n-pointed stable curve

$$(C, p_1, \ldots, p_n)$$

Since it fixes each marked point, in particular it must map each marked twig onto itself. Every twig with just one node must be a marked twig, and since the node is the only non-singular point, the node is a fixed point and that the adjacent twig (the other one attached to the node) is mapped onto itself as well. By induction on the twigs this shows that each node is a fixed point and that each twig is mapped onto itself. This shows that φ restrcited to any twig must be an automorphism of $\mathbb{P}^1$ that fixes the special points on the twig. Suppose there exists a non-trivial automorphism ψ of an arbitrary twig, then such an automorphism will also has to fix the three special points, but this will contradict Proposition 2.9, which says that there is a unique automorphism on $\mathbb{P}^1$ that moves three or more points to a a designated target. Specifically, if φ is the unique automorphism of the twig that sends three special points to $0, 1, \infty$, then $\varphi \circ \psi$ is also a automorphism that sends the three points to $0, 1, \infty$ that is distinct from φ (by assumption that ψ is non-trivial), contradicting the uniqueness of φ. Thus we conclude that φ restricted to each twig is the identity (i.e. trivial automorphism), thus φ itself is trivial. □

Enlarging the class of objects An important observation to make now is that a smooth rational curve with 3 or more marked points is stable (as an n-pointed rational curve). This directly follows from the fact that there exists a unique automorphism on $\mathbb{P}^1$ that sends three distinct points to $0, 1, \infty$ respectively. Thus for $n \geq 3$ we can consider the moduli problem with an enlarged class of objects which consists of stable n-pointed rational curves, containing n-pointed smooth rational curves as a subset. Since the equivalence relation on the objects remains the same as we defined before, we should expect that if a moduli space exists for this moduli problem, it should contain $M_{0,n}$ as a subset.

Indeed this turns out to be the case; but more is obtained from this particular enlargement of the class of object. We actually get a fine moduli space $\overline{M}_{0,n}$ for this (larger) moduli problem that is in fact the compactification of $M_{0,n}$, i.e. $M_{0,n}$ is contained as a dense subset of $\overline{M}_{0,n}$. This fact is established by a theorem by Knudsen. In essence the enlargement of the class of objects to stable rational curves is the ideal augmentation to the moduli problem to compactify $M_{0,n}$.

Before we state the existence of $\overline{M}_{0,n}$, we must formally state the moduli problem by defining the notion of families of stable n-pointed rational curves and the equivalence relation on them. The almost identical definitions with those for pointed smooth rational curves should not come as a surprise. In particular we must make sure that when restricted to smooth cases (i.e. if we take these definitions but only consider smooth n-pointed rational curves), we get exactly the same definitions as those in the previous section.

Definition 2.27 A *family of stable n-pointed curves* (over a base scheme B) is a flat and proper morphism $\pi : X \to B$ equipped with n disjoint sections, such that every geometric fiber $X_b = \pi^{-1}(b)$ is isomorphic to a stable n-pointed curve. In particular the sections are disjoint from the singular points of the fibers.

Definition 2.28 Two families of stable n-pointed curves over a common base scheme B:

$$\pi : X \to B \text{ with sections } \sigma_i, \text{ and } \pi' : X' \to B \text{ with sections } \sigma_i'$$

are said to be *equivalent* or *isomorphic* if there exists an isomorphism of n-pointed rational curves $\varphi : X \xrightarrow{\sim} X'$ such that the following diagram commutes for each $i = 1, \ldots, n$:

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & X' \\ {\scriptstyle\pi}\downarrow\uparrow{\scriptstyle\sigma_i} & & {\scriptstyle\pi'}\downarrow\uparrow{\scriptstyle\sigma_i'} \\ B & = & B \end{array}$$

The pullback operation of families along a morphism is identical to the case for unstable n-pointed curves.

And now we state the theorem of the existence of $\overline{M}_{0,n}$ due to Knudsen.

Theorem 2.29 (Knudsen [10]) *For each $n \geq 3$, there is a fine moduli space $\overline{M}_{0,n}$ for classifying stable n-pointed rational nodal curves up to isomorphism. In addition, $\overline{M}_{0,n}$ is a smooth variety, and it contains the subvariety $M_{0,n}$ as an open dense subset.*

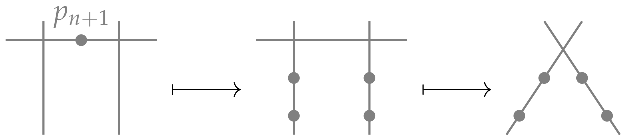
A note on more general cases It is in fact possible, more general, to construct the spaces $M_{g,n}$ and $\overline{M}_{g,n}$ which classify higher genus nodal curves, and their compactification. See [1] for detailed constructions. The compactifications are also done by imposing stability conditions. The stability condition for higher genus curves is very similar to what we have defined for genus-0 curves, with only a slight modification.

2.4 The Forgetful Map

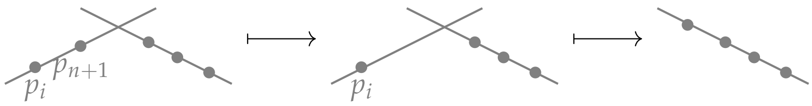
For the space $\overline{M}_{0,n+1}$, we want to find a natural map to $\overline{M}_{0,n}$ that consists of dropping one mark from each curve. Such a morphism indeed exists, but we will only give here a brief description on how it acts on the marks and twigs, and not the proof of its existence.

Consider a curve in this space, if we were to take away a mark on this curve, there is a possibility that the curve may become unstable. Thus we have to stabalize it by removing or adding twigs. First, suppose we want to forget the mark p_{n+1}, then there are two cases :

1. If p_{n+1} is on a twig without other marked points, and with just two nodes, then this twig is contracted:



2. If p_{n+1} is on a twig with just one other marked point p_i, and only one singular point (the point where the twig is attached to the rest of the curve), then the twig is contracted an the point where the twig was attached acquires the mark p_i:



Of course if dropping p_{n+1} does not affect the stability of the curve then nothing is changed other than the dropped point. The technical details of this process is encoded in the following proposition from Knudsen's paper [10], of which the proof we will not give:

Proposition 2.30 (Knudsen [10]) *Let $(X'/B, \sigma'_1, \dots, \sigma'_n, \sigma'_{n+1})$ be a family of stable $(n+1)$-pointed curves. Then there exists a family $(X/B, \sigma_1, \dots, \sigma_n)$ of stable n-pointed curves equipped with a B-morphism $\varphi : X' \to X$ such that*

1. *$\varphi \circ \sigma'_i = \sigma_i$, for $i = 1, \dots, n$;*
2. *for each $b \in B$, the induced morphism $X'_b \to X_b$ is an isomorphism when restricted to any stable twig of $(X'_b, \sigma'_1(b), \dots, \sigma'_n(b))$, and it contracts an eventual unstable twig.*

The family $(X/B, \sigma_1, \dots, \sigma_n)$ is unique up tp isomophism, and we shall say that it is the family obtained from X'/B by forgetting σ'_{n+1}. Furthermore, forgetting sections commutes with fiber products.

Our description of how twigs contract to stabalize the curve when forgetting marks is a set theoretical description of what the curve is mapped to, while the above technical proposition ensures that we actually have a morphism

$$\varepsilon : \overline{M}_{0,n+1} \to \overline{M}_{0,n}$$

which we will call the *forgetful map*.

By composing forgetful maps, we can also drop as many points as we want.

2.5 The Boundary of $\overline{M}_{0,n}$

The fact that the fine moduli space $\overline{M}_{0,n}$ contains $M_{0,n}$ as a dense subset, i.e. $\overline{M}_{0,n}$ is a compactification of $M_{0,n}$ is central to our purposes in proving Kontsevich's formula. It allows us to apply intersection theory properly on this compactified space. To that end, it is important that we study the boundary $\overline{M}_{0,n} \setminus M_{0,n}$.

For convenience we will refer to the points in $\overline{M}_{0,n}$ as curves, rather than the more precise "equivalence classes of curves". It should be clear from context we are speaking of an equivalence class, rather than a specific curve.

The next proposition gives us the description of the boundary. Consider our construction of the class of objects, which are stable n-pointed rational curves. They are trees of projective lines with additional conditions on the marked points. Now the collection of the equivalence classes of the objects $\overline{M}_{0,n}$ contains $M_{0,n}$, which consists of curves which are n-pointed smooth rational curves, i.e. irreducible curves. As such, the boundary $\overline{M}_{0,n} \setminus M_{0,n}$ should consist of only reducible stable n-pointed rational curves:

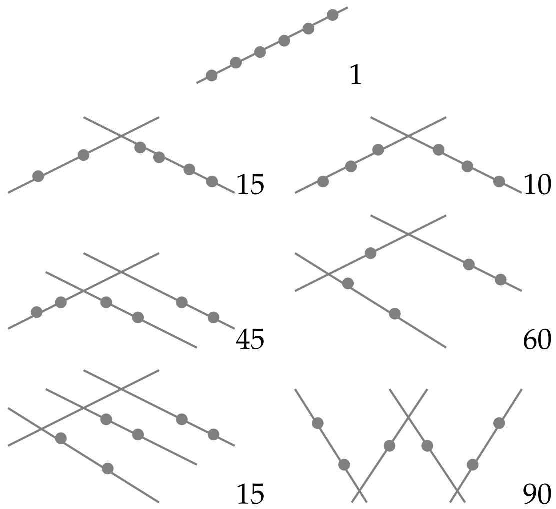
Proposition 2.31 *Each point in the boundary of $\overline{M}_{0,n}$ is a reducible stable n-pointed curve. Conversely, any reducible stable n-pointed rational curve is in the boundary.*

Proof Any stable n-pointed curve C that is irreducible must be isomorphic to $\mathbb{P}^1$ by our definition of tree of projective lines. Therefore C must be an n-pointed smooth rational curve, i.e. $C \in M_{0,n}$. Thus any point in the boundary is reducible. Conversely, every point in $M_{0,n}$ is an (irreducible) stable n-pointed smooth rational curve, which is not isomorphic to any reducible curve, Therefore any reducible curve is in the boundary. □

For a fixed number n of marked points, due to the stability condition imposed on the possible placement of marked points, there are only a finite number of configurations of stable n-pointed rational curves in. The key is that if there are too many twigs, then there will not be enough marked points to ensure each twig has at least three marked points. Consequently, there is a finite number of ways the twigs can be put together, plus a finite number of ways to permute the marked points to get a combined finite number of so-called "stratifications" of the space $\overline{M}_{0,n}$. In particular the points of the boundary $\overline{M}_{0,n} \setminus M_{0,n}$ are the stratifications which consist of reducible curves.

This notion is best illustrated by an example:

Example 2.32 The following are the possible configurations of 6-pointed stable rational curves. The number next to each diagram indicate the number of ways to label the marked points



Definition 2.33 We will call the (Zariski) closure of each stratification (i.e. each collection of curves/each diagram in Example 2.32) a *boundary cycle*. The name is chosen to reflect the fact that it has the structure of a subvariety, thus it can be trivially viewed as a formal linear combination of subvarieties (with one component and coefficient 1).

We will call the bonudary cycles with codimension 1 *boundary divisors*.

The terminology comes from intersection theory.

Proposition 2.34 *The subset Σ_δ of $\overline{M}_{0,n}$ consisting of curves with $\delta \leq n-3$ nodes is of pure dimension $n-3-\delta$.*

We will delay the proof of this proposition until the end of this section.

Thus in the above example, there is one stratum of dimension 3, which coincides with the dense subvariety $M_{0,6} \subset \overline{M}_{0,6}$; 25 strata in dimension 2; 105 strata in dimension 1; and 105 strata in dimension 0.

As a consequence of the description of how limit points behave in $\overline{M}_{0,n}$ which we will not go into detail, the boundary of a boundary cycle consists of boundary cycles of higher codimension, corresponding to configurations that have more reducible components.

Thus a general point in a boundary divisor, which are boundary cycles of codimension 1, should be a represented by a curve with two irreducible components.

Let $S = \{p_1, \ldots, p_n\}$ be the marking set. We call $S = A \cup B$ a *partition* of S, where $|A|, |B| \geq 2$ and A, B disjoint.

For each partition $S = A \cup B$, we can associate an irreducible boundary divisor $D(A|B)$. A general point on $D(A|B)$ is a curve with two twigs, with marks in A on one twig, and marks in B on the other twig.

That $D(A|B)$ is indeed an irreducible and smooth variety is established in Proposition 2.35 below.

Pulling back boundary divisor under forgetful maps Consider the forgetful map

$$\varepsilon : \overline{M}_{0,n+1} \to \overline{M}_{0,n}$$

and consider a boundary divisor $D(A|B)$ from a partition $A \cup B = S = \{p_1, \ldots, p_n\}$ in $\overline{M}_{0,n}$. The inverse image consists of two possibilities: the extra mark q can be on the A-twig, or it is on the B-twig. Thus the pullback divisor is the following

$$\varepsilon^* D(A|B) = D(A \cup \{q\}|B) + D(A|B \cup \{q\})$$

(see Lemma 1.7.1 of [5]). We will omit the justification that the coefficients are 1. This involves a local description of the forgetful map given by Knudsen.

The following proposition establishes the description of the boundary divisor $D(A|B))$. The result implies it is the product of the moduli spaces classifying curves with points on the A-twig and B-twig, respectively. In addition, this also shows that $D(A|B)$ is and irreducible and smooth variety of codimension 1.

Proposition 2.35 *Let $S = A \cup B$ be a parition and x be an additional mark. Then there is an isomorphism*

$$D(A|B) \cong \overline{M}_{0,A\cup\{x\}} \times \overline{M}_{0,B\cup\{x\}}$$

In particular, by the result of Theorem 2.29 on moduli spaces of stable curves, we can conclude that $D(A|B)$ is irreducible and smooth.

Proof A general point of $D(A|B)$ is a reducible curve with two twigs, with marks of A on one twig (call it the "A-twig"), and marks of B on the other twig (call it the "B-twig"). Place the mark x on the intersection point of the two twigs. Then the A-twig is an element of $\overline{M}_{o,A\cup\{x\}}$ and the B-twig is an element of $\overline{M}_{o,A\cup\{x\}}$. The stability condition of the twigs is equivalent to the stability condition of the curve. Conversely, given an element in $\overline{M}_{o,A\cup\{x\}} \times \overline{M}_{o,B\cup\{x\}}$, we obtain a curve of the same configuration as before, identifying the point x by attaching the two segments at this point. □

Justifying dimension count of Proposition 2.34 We generalize this argument to describe any boundary cycle as a product of moduli spaces $\overline{M}_{o,k_i}$ where the k_i's are associated to each twig.

Proof (PROOF OF PROPOSITION 2.34) We count the dimension by summing up the degrees of freedom of each twig, i.e. the freedom of moving marked points and nodes. First, A curve with δ nodes has $\delta + 1$ twigs. The curve also has $n + 2\delta$ points, where we double the δ since each node is a special point on each of the two twigs that intersect it. The stability condition ensures that there is at least three special points on each twig, and there exists an automorphism sending these three points to $0, 1, \infty$. So three of these points on the twig is spent for the automorphism, any remaining special point will contribute an additional dimension. Thus in total, we have

$$\dim \Sigma_\delta = n + 2\delta - 3(\delta + 1) = n - 3 - \delta \qquad \square$$

Chapter 3

Stable Maps and their Moduli Spaces

This chapter is concerned with the moduli problem of classifying morphisms from a projective smooth rational curve to $\mathbb{P}^r$ for some fixed $r \geq 1$.

3.1 Maps $\mathbb{P}^1 \to \mathbb{P}^r$

Definition 3.1 Let $\mu : \mathbb{P}^1 \to \mathbb{P}^r$ be a morphism. We say that μ is of *degree d* if the local expression of μ is given by

$$\mu \colon \mathbb{P}^1 \to \mathbb{P}^r$$
$$[x_0 : x_1] \mapsto [y_0 : \cdots : y_n]$$

where each y_i is a homogeneous polynomial of degree d:

$$y_i = \sum_{j=0}^{d} a_{ij} x_0^j x_1^{d-j}$$

Subject to the condition that the y_i's do not vanish simultaneously at any given point.

3.1.1 The Fine Moduli Space Classifying Maps of Degree d

According to Definition 3.1, to give a map $\mu \colon \mathbb{P}^1 \to \mathbb{P}^r$ of degree d is to specify up to a constant factor, $r+1$ binary forms (homogeneous polynomials in two variables) of degree d, which are not allowed to vanish simultaneously at any point. This condition defines a subset

$$W(r,d) \subset \mathbb{P}\left(\bigoplus_{i=0}^{r} H^0(\mathcal{O}_{\mathbb{P}^1}(d))\right)$$

so by definition there is a bijection between $W(r,d)$ and the set of all degree d maps $\mathbb{P}^1 \to \mathbb{P}^r$. In fact $W(r,d)$ is a Zariski open subset, thus endowing it with the structure of a variety.

Lemma 3.2 *The subset $W(r,d)$ is a Zariski open subset, thus it is an algebraic variety of degree $rd + r + d$.*

Proof First, the conditions that the binary forms not vanish gives that $W(r,d)$ is Zariski open. There are $(r+1)(d+1)$ degrees of freedom in choosing the binary forms. We subtract 1 degree from this because binary forms give the same map in projective space if they differ by a constant factor. This gives us $(r+1)(d+1) - 1 = rd + r + d$. □

We would like to find a moduli space classifying maps $\mathbb{P}^1 \to \mathbb{P}^r$ of degree d up to an equivalence relation to be defined shortly. The following proposition asserts that $W(r,d)$ classifies all maps $\mathbb{P}^1 \to \mathbb{P}^r$ of degree d as a fine moduli space, where the equivalence relation is the trivial one, i.e. we classify maps up to uniqueness. Another way to think about this is that we consider every map to be its own equivalence class. This should not come as too much of a surprise, since $W(r,d)$ is defined as the collection of such maps to begin with, so the bijection with equivalence classes is clear. Furthermore, Lemma 3.2 above tells us that $W(r,d)$ has the structure of a variety. The fact that $W(r,d)$ is such a moduli space is not particularly important other than a useful fact we will use later when we construct the fine moduli space classifying maps up to a non-trivial equivalence relation on the map. in which we case we will make use of its universal family.

To begin with, we define the proper notion of a family for this moduli problem as the following:

Definition 3.3 A *family of maps of smooth rational curves of degree d* (over a base scheme B with structure map π) is a diagram:

$$\begin{array}{ccc} X & \xrightarrow{\mu} & \mathbb{P}^r \\ {\scriptstyle \pi}\downarrow & & \\ B & & \end{array}$$

such that for each $b \in B$, the fiber $C_b := \pi^{-1}(b)$ is isomorphic to $\mathbb{P}^1$. Therefore the the map μ restricted to the fiber C_b:

$$\mu|_{C_b} : X \to \mathbb{P}^r$$

is a map from $\mathbb{P}^1$ to $\mathbb{P}^r$. We also impose that this map must be of degree d. In fact, it can be shown that all μ_b must have the same degree, but we will not give the proof here.

To specify a family defined this way is equivalent to defining a morphism

$$X \to B \times \mathbb{P}^r$$

This can be easily seen from the diagram. We shall use these two conventions interchangeably. Note that this morphism is not the structure map of the family; the structure map is π, which is the first factor of the above morphism, while μ is the "extra structure on the structure map π.

Eqiuvalence of families Since each of the objects we are classifying is of a distinct equivalence class, the equivalence of families is the obvious one: any two families over a common base scheme B:

$$X \xrightarrow{(\pi,\mu)} B \times \mathbb{P}^r$$

and

$$X' \xrightarrow{(\pi',\mu')} B \times \mathbb{P}^r$$

are equivalent if and only if for each $b \in B$, the following equality of maps holds:

$$\mu(\pi^{-1}(b)) = \mu'(\pi'^{-1}(b))$$

The pullback family Suppose

$$\begin{array}{ccc} X & \xrightarrow{\mu} & \mathbb{P}^r \\ {\scriptstyle\pi}\downarrow & & \\ B & & \end{array}$$

is a family, and $\varphi : B' \to B$, then the pullback of the family along φ should be the fiber product with structure map $B' \times_B X \xrightarrow{p_{B'}} B'$. Thus we have the following diagram:

$$\begin{array}{ccccc} B' \times_B X & \xrightarrow{p_X} & X & \xrightarrow{\mu} & \mathbb{P}^r \\ \downarrow{\scriptstyle p_{B'}} & & \downarrow{\scriptstyle\pi} & & \\ B' & \xrightarrow{\varphi} & B & & \end{array}$$

then the extra structure on the structure map $p_{B'}$ is given by the composition $\mu \circ p_X$. Thus, the pullback family is the following:

$$\begin{array}{ccc} B' \times_B X & \xrightarrow{\mu \circ p_X} & \mathbb{P}^r \\ {\scriptstyle p_{B'}}\downarrow & & \\ B' & & \end{array}$$

Proposition 3.4 *The space $W(r, d)$ is a fine moduli space for the moduli problem of classifying maps $\mathbb{P}^1 \to \mathbb{P}^r$ of degree d up to uniqueness.*

Proof We claim that the following family of degree d maps over $W(r,d)$ is universal:

$$\begin{array}{ccc} W(r,d)\times\mathbb{P}^1 & \xrightarrow{\sigma} & \mathbb{P}^r \\ {\scriptstyle p_W}\downarrow & & \\ W(r,d) & & \end{array}$$

where σ sends a point (γ, x) to $\gamma(x)$, where we consider $\gamma \in W(r,d)$ as a degree d map $\mathbb{P}^1 \to \mathbb{P}^r$. This is clearly a well defined family. We denote this family as $U : W(r,d) \times \mathbb{P}^1 \to W(r,d) \times \mathbb{P}^r$. Let

$$\begin{array}{ccc} X & \xrightarrow{\mu} & \mathbb{P}^r \\ {\scriptstyle \pi}\downarrow & & \\ B & & \end{array}$$

be an arbitrary family of maps of degree d. We denote this family as $F : X \to B \times \mathbb{P}^r$. We claim that this arbitrary family is the pullback of the family U along a unique morphism $\varphi : B \to W(r,d)$. We use the fact that any map $\mathbb{P}^1 \to \mathbb{P}^r$ of degree d is of the form described by Definition 3.1 (in fact this *is* the definition of a degree d map). Thus any map $\mu_b : C_b \to \mathbb{P}^r$ factors through $W(r,d)$ uniquely. More precisely, we have the following commutative diagram:

$$\begin{array}{ccccc} B & \xrightarrow{\pi^{-1}} & X & \xrightarrow{\mu_b} & \mathbb{P}^r \\ & & {\scriptstyle \exists! f}\downarrow & & \uparrow{\scriptstyle \sigma} \\ & & W(r,d) & \overset{i}{\hookrightarrow} & W(r,d)\times\mathbb{P}^1 \end{array} \tag{3.1}$$

where f sends each point in the smooth rational curve C_b to its local expression ℓ under μ_b. We can thus define the unique morphism as $b \mapsto C_b \mapsto \ell$. Pulling back the family U along this morphism:

$$\begin{array}{ccccc} B \times_{W(r,d)} (W(r,d)\times\mathbb{P}^1) = B\times\mathbb{P}^1 & \xrightarrow{p_{W\times\mathbb{P}^1}} & W(r,d)\times\mathbb{P}^1 & \xrightarrow{\sigma} & \mathbb{P}^r \\ \downarrow{\scriptstyle p_B} & & \downarrow{\scriptstyle p_W} & & \\ B & \xrightarrow{\varphi} & W(r,d) & & \end{array}$$

we obtain the family:

$$\begin{array}{ccc} B\times\mathbb{P}^1 & \xrightarrow{\sigma\circ p_{W\times\mathbb{P}^1}} & \mathbb{P}^r \\ {\scriptstyle p_B}\downarrow & & \\ B & & \end{array}$$

we claim that this family is equivalent to the family F. Indeed, by the universal property of the fiber product, the morphism $p_{W\times\mathbb{P}^1} : B \times \mathbb{P}^1 \to W(r,d) \times \mathbb{P}^1$ is the following mapping

$$(b, p) \mapsto (\varphi(b), p)$$

Therefore for any point b of the base scheme B, we have

$$\begin{aligned} \sigma \circ p_{W\times\mathbb{P}^1}\big|_{p_B^{-1}(b)} : & b \times \mathbb{P}^1 \to \mathbb{P}^r & (3.2) \\ & (b, p) \xmapsto{p_{W\times\mathbb{P}^1}} (\varphi(b), p) \overset{\sigma}{\mapsto} \varphi(b)(p) & (3.3) \end{aligned}$$

which coincides with the map μ_b according to the diagram 3.1. This establishes the desired equivalence of families. □

Drawbacks of $W(r,d)$ For classifying maps from a smooth rational curve to $\mathbb{P}^r$, the moduli space $W(r,d)$ is inadequate for our purposes. For one thing, it is not true that every family of rational curves admits a family of parametrization from one and the same $\mathbb{P}^1$. Another problem is redundancy: reparametrizations of the same rational curve in $\mathbb{P}^r$ are considered distinct objects.

3.1.2 The Fine Moduli Space Classifying Maps of Degree d up to Isomorphism

The space $W(r,d)$ introduced in the previous section is insufficient since it admits different parametrizations for maps that are from a "different $\mathbb{P}^1$", that is to say, the space $W(r,d)$ is sensitive to isomoprhisms of $\mathbb{P}^1$. To be more precise, we would like to classify our objects, which are degree d maps $\mathbb{P}^1 \to \mathbb{P}^r$ up to an isomorphism that is given by an isomorphism on the domain $\mathbb{P}^1$. This will give us a moduli problem of classifying maps $C \to \mathbb{P}^r$ of degree d, where C is a projective smooth rational curve, up to isomorphism of maps.

Definition 3.5 Let $\mu : C \to \mathbb{P}^r$ and $\mu' : C' \to \mathbb{P}^r$ be morphisms, where C, C' are projective smooth rational curves. An *isomoprhism* between the maps μ and μ' is an isomorphism of projective smooth rational curves $\varphi : C \to C'$ such that the following diagram commutes:

$$\begin{array}{ccc} C & \xrightarrow{\varphi} & C' \\ & {\scriptstyle\mu}\searrow \quad \swarrow{\scriptstyle\mu'} & \\ & \mathbb{P}^r & \end{array}$$

However, by Proposition 2.19, we know that all projective smooth rational curves are isomorphic to $\mathbb{P}^1$ Therefore this definition actually just imposes that two maps are isomorphic if they are the same map up to an automorphism on the domain $\mathbb{P}^1$.

The notion of equivalence of families will be the following:

Definition 3.6 Given families (of maps of smooth rational curves) $F : X \xrightarrow{(\pi,\mu)} B \times \mathbb{P}^r$ and $F' : X' \xrightarrow{(\pi',\mu')} B \times \mathbb{P}^r$ over a common base scheme B, we say they are *equivalent* if for any $b \in B$, the following maps are equivalent in the sense of Definition 3.5:

$$\mu \circ \pi^{-1}(b) \sim \mu' \circ \pi'^{-1}(b)$$

In view of this notion of equivalence of families, we should expect that if a moduli space (fine or coarse) exists for the moduli problem of classifying maps $C \to \mathbb{P}^1$ of degree d, where C is a projective smooth rational curve up to isomorphism of maps, the space shoould to be set theoretically bijective with the quotient set

$$W(r,d) \Big/ \mathrm{Aut}(\mathbb{P}^1)$$

Plausibility of existence of a moduli space It is a general fact that when classifying algebro-geometric objects up to a certain equivalence, if every object is automorphism-free, then the existence of a fine moduli space can be expected. If each object as a finite group of automorphisms, then the existence of a coarse moduli space can be expected. Lastly, in general, if some object has an infinite automoprhism group, then the existence of either fine or coarse moduli space cannot be expected. These should not be taken as direct proofs of the existence of these spaces however, as there are exceptions to this rule. They instead serve as indicators to motivate or discourage one from movinf forward in the search for them.

The following proposition suggests at least the plausibility of the existence of a course moduli space classifying maps of degree d up to isomorphism of maps.

First, we recall a result that relates maps between function fields of two varieties and rational maps between the varieties.

Lemma 3.7 (Hartshorne [7] Theorem 4.4) *For any two quasi-projective varieties X and Y over an algebraically closed field k, there is a bijection between*

1. *the set of dominant rational maps from X to Y, and*
2. *the set of k-algebra homomorphisms from $K(Y)$ to $K(X)$.*

Furthermore, this correspondence gives a arrow-reversing equivalence of categories of the category of varieties and dominant rational maps with the category of finitely generated field extensions of k.

Proposition 3.8 *Let $\mu : \mathbb{P}^1 \to \mathbb{P}^r$ be a nonconstant map. Then there is only a finite number of automorphisms $\varphi : \mathbb{P}^1 \to \mathbb{P}^1$ such that $\mu = \mu \circ \varphi$. If μ is birational onto its image, then* $\mathrm{Aut}(\mu)$ *is trivial.*

Proof Let $\mathbb{C}(\mu(\mathbb{P}^1))$ be the function field of the image curve $\mu(\mathbb{P}^1) \subset \mathbb{P}^r$, and let $\mathbb{C}(\mathbb{P}^1)$ be the function field of $\mathbb{P}^1$. Then $\mathbb{C}(\mathbb{P}^1)/\mathbb{C}(\mu(\mathbb{P}^1))$ is a finite field extension. In general the function field of a quasi-affine variety Y over an algebraically closed field k is a finitely generated field extension over k, as $K(Y) = K(U)$ for any U open affine subset, we may assume Y is affine, so $K(Y)$ is isomorphic to the quotient field of the coordinate ring, which is finitely generated. On the other hand, the

Therefore the field extension $\mathbb{C}(\mu(\mathbb{P}^1)) \hookrightarrow \mathbb{C}(\mathbb{P}^1)$ is a finite field extension. Now by Lemma 3.7 there is a bijection between the automorphism group of μ with the group of automorphisms of $\mathbb{C}(\mathbb{P}^1)$ that fixes the subfield $\mathbb{C}(\mu(\mathbb{P}^1))$. Since the field extension is finite, this group is finite as well, thus the desired result. The condition that μ is birational onto its image is equivalent to $\mathbb{C}(\mu(\mathbb{P}^1)) = \mathbb{C}(\mathbb{P}^1)$, which then implies the automorphism group of μ is trivial. □

It is in fact true that there exists a coarse moduli space classifying maps up to isomorphism. We will however not pursue this line of inquiry further.

3.2 Pointed Maps

As with the case of the fine moduli space M classifying curves The goal of this section is to add the additional structure of marked points to the source curve of a map classification of maps from n-pointed projective smooth rational curves to $\mathbb{P}^r$, up to an equivalence that is given by an isomorphism of curves which respects both the marked points, and the respective mappings.

The objects we will be classifying are maps $C \to \mathbb{P}^r$ of degree d up to isomorphism, where C is an n-pointed tree of projective lines. Isomorphism here refers to an isomorphism of curves that respects the marked points.

Definition 3.9 An *n-pointed map* is a morphism $\mu : C \to \mathbb{P}^r$, where

$$(C, p_1, \dots, p_n)$$

is a nodal curve with n distinct marked points that are smooth points of C. We will often denote an n-pointed map as

$$(C, p_1, \dots, p_n; \mu)$$

Note that at the moment we have not imposed any sort of stability conditions, thus the marked points are free to be placed on any twig, and there are no constraints on the number of marked points on a given twig (except of course the total number of marked points must be n). An isomorphism of n-pointed maps is an isomoprhism of curves that respects all structures:

Definition 3.10 An isomoprhism of n-pointed maps $\mu : C \to \mathbb{P}^r$ and $\mu' : C' \to \mathbb{P}^r$ is an isomorphism of source curves $\varphi : C \xrightarrow{\sim} C'$ making the following diagrams commute:

$$\begin{array}{ccc} C & \xrightarrow{\varphi} & C' \\ & \searrow^{\mu} \quad \swarrow_{\mu'} & \\ & \mathbb{P}^r & \end{array}$$

and

$$\begin{array}{ccc} C & \xrightarrow{\varphi} & C' \\ \pi \downarrow \uparrow \sigma_i & & \pi' \downarrow \uparrow \sigma_i' \\ \bullet & = & \bullet \end{array}$$

where σ_i, σ_i' single out the marked points in the source curve.

Families of objects will be the following:

Definition 3.11 A *family of n-pointed maps of degree d* (over a base scheme B with structure map π) is a diagram:

$$\begin{array}{ccc} X & \xrightarrow{\mu} & \mathbb{P}^r \\ \pi \downarrow \uparrow \sigma_i & & \\ B & & \end{array}$$

where π is a flat morphism such that each geometric fiber $C_b := \pi^{-1}(b)$ is isomorphic to a tree of projective lines. Thus π can be thought of as a family of trees of projective lines. The σ_i are n disjoint sections that do not meet the singular points of the fibers of π, i.e. over any point $b \in B$, $\sigma(b) \in \pi^{-1}(b)$ is non-singular.

Therefore the the map μ restricted to the fiber C_b:

$$\mu|_{C_b} : X \to \mathbb{P}^r$$

is a map from an n-pointed tree of projective lines C_b to $\mathbb{P}^r$. We also impose that this map must be of degree d. In fact, it can be shown that all μ_b must have the same degree.

To specify a family defined this way is equivalent to defining a morphism

$$\xi : X \to B \times \mathbb{P}^r$$

that respects the sections, i.e.

$$p_B \circ \xi \circ \sigma_i = \pi \circ \sigma_i = \mathrm{id}_B$$

for every i. We shall use these two descriptions interchangeably.

The notion of equivalence of families will be similar to the case for non-pointed maps, except the isomorphism will be given by isomorphism of n-pointed curves.

Definition 3.12 Given families of maps of n-pointed curves of degree d $F : X \xrightarrow{(\pi,\mu)} B \times \mathbb{P}^r$ with sections σ_i and $F' : X' \xrightarrow{(\pi',\mu')} B \times \mathbb{P}^r$ with sections σ_i' over a common base scheme B, we say they are *equivalent* if for any $b \in B$, the following maps are isomorphic in the sense of Definition 3.10:

$$\mu\big|_{\pi^{-1}(b)} \cong \mu'\big|_{\pi'^{-1}(b)}$$

and unsurprisingly:

Definition 3.13 Suppose $F : X \xrightarrow{(\pi,\mu)} B \times \mathbb{P}^r$ is a family of maps of n-pointed curves of degree d, and $\varphi : B' \to B$ a morphism. Then the *pullback of F along* φ is the family

$$\begin{array}{ccc} B' \times_B X & \xrightarrow{p_X \circ \mu} & \mathbb{P}^r \\ \pi \big\downarrow \big\uparrow \zeta_i & & \\ B' & & \end{array}$$

where the ζ_i's are the unique maps from B' ro $B' \times_B X$ given by the universal property of the fiber product.

Now for the main takeaway for this section, which is the existence of the fine moduli space classifying pointed maps of degree d.

Theorem 3.14 *For each $n \geq 3$ there is a fine moduli space $M_{0,n}(\mathbb{P}^r, d)$ classifying n-pointed maps $\mathbb{P}^1 \to \mathbb{P}^r$ of degree d, namely*

$$M_{0,n}(\mathbb{P}^r, d) = M_{0,n} \times W(r, d)$$

Proof We claim that the following family has the universal property:

$$\begin{array}{ccc} M_{0,n} \times W(r,d) \times \mathbb{P}^1 & \xrightarrow{\sigma} & \mathbb{P}^r \\ p_{M \times W} \big\downarrow \big\uparrow \sigma_i & & \\ M_{0,n} \times W(r,d) & & \end{array}$$

where the sections σ_i are the sections of the universal family of $M_{0,n}$, in particular the first three sections are the constant ones $0, 1, \infty$, and the rest are projections, see Theorem 2.13 for details. The morphism σ is identical to the one in the universal family over $W(r,d)$ of Proposision 3.4, except the source has an additional factor from $M_{0,n}$. To be precise, the map is

$$\begin{aligned}\sigma\colon M_{0,n} \times W(r,d) \times \mathbb{P}^1 &\to \mathbb{P}^r \\ (q,\gamma,p) &\mapsto \gamma(p)\end{aligned}$$

where we interpret the point γ as a map $\mathbb{P}^1 \to \mathbb{P}^r$.

Now let

$$\begin{array}{ccc} X & \xrightarrow{\mu} & \mathbb{P}^r \\ {\scriptstyle\pi}\big\downarrow\big\uparrow{\scriptstyle\rho_i} & & \\ B & & \end{array}$$

be an arbitrary family of maps of n-pointed smooth rational curves. We aim to show the existence of a unique morphism $\Phi : B \to M_{0,n} \times W(r,d)$ such that the arbitrary family is isomorphic to the pullback of the claimed universal family. Notice that

$$\begin{array}{c} X \\ {\scriptstyle\pi}\big\downarrow\big\uparrow{\scriptstyle\rho_i} \\ B \end{array}$$

is a family of n-pointed projective smooth rational curves. In particular, by Proposition 2.20, there is a unique isomorphism $\varphi : X \to B \times \mathbb{P}^1$ making this family of pointed curves isomorphic to the trivial family

$$\begin{array}{c} B \times \mathbb{P}^1 \\ {\scriptstyle p_B}\big\downarrow\big\uparrow{\scriptstyle\gamma_i} \\ B \end{array}$$

where the first three sections send a point to $0, 1, \infty$ in $\mathbb{P}^1$ respectively. By the universal property of the fine moduli space $M_{0,n}$, there exists a unique morphism $B \to M_{0,n}$ inducing this trivial family from the universal family of n-pointed smooth rational curves, i.e.

$$\begin{array}{c} M_{0,n} \times \mathbb{P}^1 \\ {\scriptstyle p_M}\big\downarrow\big\uparrow{\scriptstyle\sigma_i} \\ M_{0,n} \end{array}$$

On the other hand, the universal property of $W(r,d)$ ensures that our family $B \times \mathbb{P}^1 \to \mathbb{P}^r$ is induced by the universal family $W(r,d) \times \mathbb{P}^1 \to \mathbb{P}^r$ via a unique morphism $B \to W(r,d)$. Combining the two unique morphisms we obtain the morphism $\Phi : B \to M_{0,n} \times W(r,d)$. Pullingback the claimed universal family along this morphism:

$$\begin{array}{ccccc} B \times_{M_{0,n}\times W(r,d)} M_{0,n} \times W(r,d) \times \mathbb{P}^1 & \xrightarrow{p_{M\times W\times \mathbb{P}^1}} & M_{0,n} \times W(r,d) \times \mathbb{P}^1 & \xrightarrow{\sigma} & \mathbb{P}^r \\ p_B \downarrow \uparrow \zeta_i & & p_{M\times W} \downarrow \uparrow \sigma_i & & \\ B & \xrightarrow{\Phi} & M_{0,n} \times W(r,d) & & \end{array}$$

reduces down to give us the family

$$\begin{array}{ccc} B \times \mathbb{P}^1 & \xrightarrow{\sigma \circ p} & \mathbb{P}^r \\ p_B \downarrow \uparrow \zeta_i & & \\ B & & \end{array}$$

where $p = p_{M\times W\times \mathbb{P}^1}$, and ζ_i are the unique sections on the pullback family given by the universal property of the fiber product. It remains to show that this pullback family is equivalent to the arbitrary family.

By the commutativity of the above diagram, the morphism p must be

$$p(b,p) = \Phi(b) \times p$$

Since we know that the arbitrary family is equivalent to the trivial family

$$\begin{array}{ccc} B \times \mathbb{P}^1 & \xrightarrow{\mu} & \mathbb{P}^r \\ p_B \downarrow \uparrow \gamma_i & & \\ B & & \end{array}$$

it suffices to show that $\zeta_i = \gamma_i$ for any i. □

3.3 Compactifying $M_{0,n}(\mathbb{P}^r, d)$ via Stable Maps

In a similar fashion to the process of compactifying $M_{0,n}$ to obtain $\overline{M}_{0,n}$, we will compactify the space $M_{0,n}(\mathbb{P}^r, d)$ by enlarging the class of objects to the collection of *n-pointed stable maps*, which allow the source curve to be a tree of projective lines with n-marked points, subject to a stability condition on the map.

Definition 3.15 An n-pointed map $\mu\colon (C, p_1, \ldots, p_n) \to \mathbb{P}^r$ is called *stable* if any twig which is mapped to a point is stable as a pointed rational curve; i.e. there is at least three special points on it. See Definition 2.25 for details.

It is important to note that the source curve C of a stable n-pointed map need not be stable. For example, consider a non-constant map $\mathbb{P}^1 \to \mathbb{P}^r$.

There is only one twig on $\mathbb{P}^1$, which is $\mathbb{P}^1$ itself; it cannot be mapped to a point since the map is non-constant. Thus the condition to be an n-pointed stable map for any n is vacuously satisfied. If we take $n = 0$ then $\mathbb{P}^1$ with no marked points is obviously not stable as a rational curve.

The equivalence relation on these objects is almost exactly the same for n-pointed maps of smooth rational curves:

Definition 3.16 An *isomoprhism* of stable n-pointed maps $\mu : (C, p_1, \dots, p_n) \to \mathbb{P}^r$ and $\mu' : (C', p_1;, \dots, p'_n) \to \mathbb{P}^r$ is an isomorphism of source curves $\varphi : C \xrightarrow{\sim} C'$ making the following diagrams commute:

$$\begin{array}{ccc} C & \xrightarrow{\varphi} & C' \\ & {}_{\mu}\searrow \quad \swarrow_{\mu'} & \\ & \mathbb{P}^r & \end{array}$$

and

$$\begin{array}{ccc} C & \xrightarrow{\varphi} & C' \\ {\scriptstyle \pi}\downarrow\uparrow{\scriptstyle \sigma_i} & & {\scriptstyle \pi'}\downarrow\uparrow{\scriptstyle \sigma'_i} \\ \bullet & = & \bullet \end{array}$$

where σ_i, σ'_i single out the marked points in the source curve.

Now we define a *family of stable n-pointed maps*, which is just a slight generalization

The following lemma characterizes stable n-pointed maps, and also indicates that a coarse moduli space should exist.

Lemma 3.17 *An n-pointed map is stable if and only if it has a finite number of automorphism.*

Proof Let $\mu : (C, p_1, \dots, p_n) \to \mathbb{P}^r$ be a stable map. If the source curve is stable as an n-pointed rational curve, then there are no automorphisms by Proposition 2.26. If the source curve is not stable, then there exists some twig E of the source curve that is unstable as an n-pointed rational curve. Thus by the stability of the map μ, E is not mapped to a point, which means $\mu|_E$ is a non-constant map $\mathbb{P}^1 \to \mathbb{P}^r$. Now let φ be an automorphism of the map μ, and let $E' := \varphi(E)$, then $\mu|_{E'} \circ \varphi|_E = \mu|_E$. Now Proposition 3.8 implies that there are only finitely many automorphisms of $\mu|_E$. Thus an infinite number of automorphisms φ will lead to a contradiction.

Conversely, suppose $\mu : (C, p_1, \dots, p_n) \to \mathbb{P}^r$ is an unstable map. Then there is an unstable twig E that is mapped to a point under μ. By Proposition 2.26, a curve being unstable is equivalent to it having infinite automorphisms. Each one of these automorphism of E can be extended to the whole of C by imposing the identity on the other twigs. Since the image $\mu(E)$ is a point, these automorphisms commute with μ, satisfying the condition for them to be automorphisms of the map μ. Therefore μ has infinite automorphisms.□

Remark 3.18 *Recall from the previous section we constructed the fine moduli space $M_{0,n}(\mathbb{P}^r, d)$ classifying n-pointed maps $\mathbb{P}^1 \to \mathbb{P}^r$ of degree d, for $n \geq 3$. Since we know that for $n \geq 3$, there are no non-trivial automorphisms of $\mathbb{P}^1$, thus first and foremost all maps in $M_{0,n}(\mathbb{P}^r, d)$ are stable.*

Now we can state the following two existence theorems of the space, essentially distilled from more general results found in [6]. We will not give the proofs.

Theorem 3.19 (F-P [6]) *There exists a coarse moduli space $\overline{M}_{0,n}(\mathbb{P}^r, d)$ classifying stable n-pointed maps to $\mathbb{P}^r$ of degree d up to isomoprhism.*

Theorem 3.20 (F-P [6]) *The coarse moduli space $\overline{M}_{0,n}(\mathbb{P}^r, d)$ is a projective irreducible variety, and it is locally isomorphic to a quatient of a smooth variety by the action of a finite group. It contains $M_{0,n}(\mathbb{P}^r, d)$ as a smooth open dense subvariety which is a fine moduli space for maps without automorphisms.*

3.4 Canonical Maps

There are two important canonical maps from $\overline{M}_{0,n}(\mathbb{P}^r, d)$ that will be of use to us. One of them is the forgetful map that mirrors that of $\overline{M}_{0,n}$. But first, we look at the so-called *evaluation maps*. Unfortunately, the proof of their existence, that they are morphisms, and several properties which we will need are beyond the scope of this paper, so we will omitt them. They can be found in Knudsen's paper [10]. We will only give set-theoretic descriptions for them here.

3.4.1 Evaluation Maps

For each mark p_i there is a natural map

$$\begin{aligned} \mathrm{ev}_i \colon \overline{M}_{0,n}(\mathbb{P}^r, d) &\to \mathbb{P}^r \\ [(C;, p_1, \ldots, p_n; \mu)] &\mapsto \mu(p_i) \end{aligned}$$

called the *evaluation map*. It sends a map to its value on the mark p_i. One property of the evaluation maps we will need later is that they are flat:

Lemma 3.21 *The evaluation maps are flat.*

3.4.2 Forgetful Maps

As with the case of stable curves, we can define a forgetful map:

$$\varepsilon : \overline{M}_{0,n+1}(\mathbb{P}^r, d) \to \overline{M}_{0,n}(\mathbb{P}^r, d)$$

Which enables us to define for each choice of sets of marks $B \subset A$ a map

$$\overline{M}_{0,A}(\mathbb{P}^r, d) \to \overline{M}_{0,B}(\mathbb{P}^r, d)$$

The description of how the forgetful map affects a map with reducible source curve is similar to the case for $\overline{M}_{0,n}$. Twigs that become unstable by the absence of the dropped mark must be contracted. We will omit the details here.

Next is the existence of a morphism from the moduli space of maps to the moduli space of stable curves:

Proposition 3.22 (Forgetting the map to $\mathbb{P}^r$) *For $n \geq 3$, there is a forgetful map*

$$\eta : \overline{M}_{0,n}(\mathbb{P}^r, d) \to \overline{M}_{0,n}$$

Roughly speaking, this map is constructed by forgetting the data of the map to $\mathbb{P}^r$, then stabilizing the source curve. This map is also (confusingly) called the *forgetful map* as well. Although the context should make it clear which map is being talked about.

This forgetful map η is in fact also a flat morphism:

Proposition 3.23 *For $n \geq 3$, the forgetful map*

$$\eta : \overline{M}_{0,n}(\mathbb{P}^r, d) \to \overline{M}_{0,n}$$

is a flat morphism.

3.5 The Boundary of $\overline{M}_{0,n}(\mathbb{P}^r, d)$

In this section, we will show that there is a combinatorial description of the boundary of $\overline{M}_{0,n}(\mathbb{P}^r, d)$. This will be central to our proof of Kontsevich's formula, where this combinatorial description will lead to the very one that gives the recursive relation in the formula.

The boundary of $\overline{M}_{0,n}(\mathbb{P}^r, d)$ resembles that of $\overline{M}_{0,n}$, it consists of maps that have reducible curves as their domains.

Lemma 3.24 *The boundary $\overline{M}_{0,n}(\mathbb{P}^r, d) \setminus M_{0,n}(\mathbb{P}^r, d)$ consists of all the maps whose source curves, i.e. domains, are reducible curves.*

Proof If $(C; p_1, \dots, p_n; \mu)$ is an n-pointed stable map $\mu : C \to \mathbb{P}^r$ where C is an irreducible tree of projective lines, thus is isomorphic to $\mathbb{P}^1$, then the only automorphism of μ is the identity. Therefore $(C; p_1, \dots, p_n; \mu) \in \overline{M}^*_{0,n}(\mathbb{P}^r, d)$. Conversely, suppose $(C; p_1, \dots, p_n; \mu)$ is an n-pointed stable map that is automorphism free. Then if C is reducible □

Definition 3.25 Let d be a non-negative integer. A *d-weighted partition* of a set $S := \{p_1, \dots, p_n\}$ consists of a partition $A \cup B = S$ of S and a partition $d_A + d_B = d$ of d, where d_A and d_B are non-negative integers.

We can carry over the descriptions of the boundary on $\overline{M}_{0,n}$ to the case of maps, where boundary cycles are maps with reducible source. We must also be mindful of the degree of the maps however. We will need the following lemma:

Lemma 3.26 *Let $d \geq 2$ be an integer. For each d-weighted partition*

$$A \cup B = S, \quad d_A + d_B = d$$

there exists an irreducible divisor $D(A, B; d_A, d_B)$ called a boundary divisor, *where a general point on this divisor is a map μ whose domain is a nodal curve C with two twigs C_A and C_B, with points of A on C_A and those of B on C_B, such that the restriction of μ to C_A is a map of degree d_A and the restriction of μ to C_B is a map of degree d_B.*

We also have an analogue to Lemma 2.35:

Lemma 3.27 (F-P [6] Lemma 12) *There is an isomorphism*

$$\overline{M}_{0,A\cup\{x\}}(\mathbb{P}^r, d_A) \times_{\mathbb{P}^r} \overline{M}_{0,B\cup\{x\}}(\mathbb{P}^r, d_B) \xrightarrow{\sim} D(A, B; d_A, d_B)$$

Here is the combinatorial description, we we call the *fundamental relation*.

Theorem 3.28 (fundamental relation) *Let $n \geq 4$. Let*

$$\eta : \overline{M}_{0,n}(\mathbb{P}^r, d) \to \overline{M}_{0,n} \to \overline{M}_{0,4}$$

be the composition of forgetful maps. Let $D^{-1}(ij|kl) := \eta^{-1}(D(ij|kl))$ where $D(ij, kl)$ is a divisor in $\overline{M}_{0,4}$ (so in particular $S = \{i, j, k, l\}$ are the four marks). Then

$$D^{-1}(ij|kl) = \sum D(A, B; d_A, d_B) \tag{3.4}$$

where the sum is taken over all d-weighted partitions of the markings $S = \{p_1, \dots, p_n\}$ such that $i, j \in A$ and $k, l \in B$. Furthermore, we have the fundamental relation:

$$\sum_{\substack{A\cup B=S\\ i,j\in A\\ k,l\in B\\ d_A+d_B=d}} D(A, B; d_A, d_B) \equiv \sum_{\substack{A\cup B=S\\ i,k\in A\\ j,l\in B\\ d_A+d_B=d}} D(A, B; d_A, d_B) \equiv \sum_{\substack{A\cup B=S\\ i,l\in A\\ j,k\in B\\ d_A+d_B=d}} D(A, B; d_A, d_B) \tag{3.5}$$

Proof We will omit the proof that the coefficients of 3.4 are all 1. By Lemma 1.7.1 of [5], the pullback cycle is the cycle consisting of the inverse scheme under a flat morphism, and by Proposition 3.23 we know this map η is indeed flat. First, we pull back the divisor $D(ij|kl)$ to $\overline{M}_{0,n}$ to get n-pointed stable curves on two twigs, i.e. the cycle

$$\sum_{\substack{A\cup B=S\\ i,j\in A\\ k,l\in B}} D(A|B) \tag{3.6}$$

Recall that any two points on $\overline{M}_{0,\{i,j,k,l\}} = \overline{M}_{0,4} \cong \mathbb{P}^1$ are linearly equivalent, thus their pullbacks to $\overline{M}_{0,n}$ are linearly equivalent as well. In particular the pullbacks of divisors $D(ij|kl), D(ik|jl)$, and $D(il|jk)$ are linearly equivalent:

$$\sum_{\substack{A\cup B=S\\ i,j\in A\\ k,l\in B}} D(A|B) \equiv \sum_{\substack{A\cup B=S\\ i,k\in A\\ j,l\in B}} D(A|B) \equiv \sum_{\substack{A\cup B=S\\ i,l\in A\\ j,k\in B}} D(A|B) \tag{3.7}$$

Then pulling back (3.7) to $\overline{M}_{0,n}(\mathbb{P}^r, d)$ gives us the desired fundamental relation of (3.5). □

3.6 The Dimension of $\overline{M}_{0,n}(\mathbb{P}^r, d)$

The following proposition gives the dimension of $\overline{M}_{0,n}(\mathbb{P}^r, d)$ for $n \geq 3$. The proof however, will need the coarse moduli space

$$M_{0,0}(\mathbb{P}^r, d) \cong W(r,d) \Big/ \mathrm{Aut}(\mathbb{P}^1)$$

Several comments are in order: This is a geometric quotient in the sense of Mumford [13]. For our purposes however, it suffices to consider this as a quotient of complex (smooth) manifolds, and that there exists the classification morphism

$$W(r,d) \to M_{0,0}(\mathbb{P}^r, d)$$

The fact that $M_{0,0}(\mathbb{P}^r, d)$ is a coarse moduli space classifying (unpointed) maps is not substantiated, and is a rather deep result.

Proposition 3.29 *The dimension of $\overline{M}_{0,n}(\mathbb{P}^r, d)$ is*

$$\dim \overline{M}_{0,n}(\mathbb{P}^r, d) = rd + r + d + n - 3$$

Proof It suffices to show the result for the dense open subset $M_{0,n}(r,d)$, i.e. that

$$\dim M_{0,n}(r,d) = rd + r + d + n - 3$$

First consider the dimension of the unpointed case. Under the classification morphism above, the generic fiber is isomorphic to $\mathrm{Aut}(\mathbb{P}^1) = \mathrm{PGL}(2, \mathbb{C})$ (as a smooth complex manifold). Since the classification morphism is a dominant morphism of varieties, by Corollary 13.5 of [4], the dimension of $M_{0,0}(\mathbb{P}^r, d)$ is

$$\dim M_{0,0}(\mathbb{P}^r, d) = \dim W(r, d) - \dim \mathrm{Aut}(\mathbb{P}^1) = rd + r + d - 3$$

Now observe that for each additional marked point, the dimension increments by 1, thus the desired result. □

Chapter 4

Kontsevich's Formula

4.1 Transversality of Intersection

Consider the n evaluation maps $\{\mathrm{ev}_i\}_{i=1}^n$ on the moduli space of n-pointed degree d stable maps $\overline{M}_{0,n}(\mathbb{P}^r, d)$. They induce a map $\underline{\mathrm{ev}} : \overline{M}_{0,n}(\mathbb{P}^r, d) \to (\mathbb{P}^r)^n$ by simply taking taking the n-tuple of the evaluation maps. Let $\tau_i : (\mathbb{P}^r)^n \to \mathbb{P}^r$ be the i-th projection map, then we have the commutative diagram

$$\begin{array}{ccc} \overline{M}_{0,n}(\mathbb{P}^r, d) & \xrightarrow{\quad\underline{\mathrm{ev}}\quad} & (\mathbb{P}^r)^n \\ & {\scriptstyle \mathrm{ev}_i} \searrow \quad \swarrow {\scriptstyle \tau_i} & \\ & \mathbb{P}^r & \end{array}$$

Given $\Gamma_1, \ldots, \Gamma_n \subset \mathbb{P}^r$ irreducible subvarieties, denote their product:

$$\underline{\Gamma} := \Gamma_1 \times \cdots \times \Gamma_n = \bigcap_{i=1}^{n} \tau_i^{-1}(\Gamma_i) \subset (\mathbb{P}^r)^n$$

We also denote k_i the codimension of Γ_i in $\mathbb{P}^r$.

By the flatness of the evaluation maps (Lemma 3.21), the inverse images $\mathrm{ev}_i^{-1}(\Gamma_i)$, consisting of all (equivalence classes of) maps μ such that $\mu(p_i) \in \Gamma_i$, has codimension k_i in $\overline{M}_{0,n}(\mathbb{P}^r, d)$ as well.

We are interested in the case when the codimensions of the Γ_i's add up to the dimension of $\overline{M}_{0,n}(\mathbb{P}^r, d)$. In which case, we will show that the intersection

$$\mathrm{ev}_i^{-1}(\Gamma_1) \cap \cdots \cap \mathrm{ev}_n^{-1}(\Gamma_n) = \underline{\mathrm{ev}}^{-1}(\underline{\Gamma})$$

has dimension zero. In the proof of Kontsevich's formula, each Γ_i will represent an incidence condition on the curves, and having dimension zero allows us to conclude that there are finitely many curves satisfy these constraints.

Let us recall the notion of a *group variety*, for details see Hartshorne [7] Section III.10. A *group variety* G over an algebraically closed field k is a variety G, together with morphisms $\mu : G \times G \to G$ and $\rho : G \to G$, such that the set $G(k)$ of k-valued points, equivalently closed points of G becomes a group under the operation induced by μ with ρ giving the inverses.

We say that a group variety *acts* on a variety X if there is a morphism $\theta : G \times X \to X$ which induces a homomorphism of groups $G(k) \to X$.

A *homogeneous space* is a variety X together with a group variety G acting on it, such that the group $G(k)$ acts transitively on the the set $X(k)$ of closed points of X.

In particular, the projective space $\mathbb{P}^n_{\mathbb{C}}$ (and more generally over any algebraically closed k) is a homogeneous space for the action $G = \mathrm{PGL}(n)$.

Now we present a theorem by Kleiman.

Theorem 4.1 (Kleiman [9] Theorem 2. or Hartshorne [7] Theorem 10.8) *Let X be a homogeneous space with group variety G over an algebraically closed field k of characteristic 0. Let $f : Y \to X$ and $g : Z \to X$ be morphisms of irreducible varieties Y and Z to X. For any $\sigma \in G(k)$, let Y^σ denote Y considered as a variety over X via the composition, in other words Y^σ is the morphism of varieties $\sigma \circ f : Y \to X$. Then there exists a non-empty dense open subset $U \subset G$ such that for every $\sigma \in U(k)$, the fiber product $Y^\sigma \times_X Z$ is either empty or of dimension exactly*

$$\dim(Y^\sigma \times_X Z) = \dim Y + \dim Z - \dim X$$

Furthermore, if Y and Z are smooth, then U can be chosen such that for every $\sigma \in U$, the fiber product $Y^\sigma \times_X Z$ is also smooth.

Lemma 4.2 ([6] Lemma 13.) *Let $M^*_{0,n}(\mathbb{P}^r, d) \subset \overline{M}_{0,n}(\mathbb{P}^r, d)$ be the locus of maps with smooth source curve and without automorphisms. Then $M^*_{0,n}(\mathbb{P}^r, d)$ is a dense open set.*

The proof of the following result will essentially be a repeated application of this theorem for the case $k = \mathbb{C}$, $X = (\mathbb{P}^r)^n$, and $G = \mathrm{PGL}(n)^n$.

Proposition 4.3 *For generic choices of $\Gamma_1, \ldots, \Gamma_n \subset \mathbb{P}^r$, with $\sum_{i=1}^n \operatorname{codim} \Gamma_i = \dim \overline{M}_{0,n}(\mathbb{P}^r, d)$, the scheme theoretic intersection*

$$\underline{\mathrm{ev}}^{-1}(\underline{\Gamma}) = \bigcap_{i=1}^{n} \mathrm{ev}_i^{-1}(\Gamma_i)$$

consists of finite number of reduced points, supported in any preassigned non-empty open set, and in particular, in the locus $M^*_{0,n}(\mathbb{P}^r, d) \subset \overline{M}_{0,n}(\mathbb{P}^r, d)$ of maps with smooth source curve and without automorphism.

Proof By abuse of notation, we denote M^* be a chosen nonempty open set. Let $\underline{G}$ denote the product of n copies of the group $G = \mathrm{PGL}(n)$, it acts transitively on $X^n = (\mathbb{P}^r)^n$ by translation. We have the morphisms

$$i : \underline{\Gamma} \to X^n$$

which is inclusion, and the n-fold evaluation map

$$\underline{\mathrm{ev}} : (M^*)^c \to X^n$$

Let $\sigma \in G$, recall that $\underline{\Gamma}^\sigma$ denotes the variety $\underline{\Gamma}$ considered as a variety over X via the composition $\sigma \circ f$; in other words, it is a translation of $\underline{\Gamma}$ by an element σ of $\underline{G}$. The inverse image of $\underline{\Gamma}^\sigma$ under the map $\underline{\nu}$ is identified with the fiber product $\underline{\Gamma}^\sigma \times_{X^n} (M^*)^c$. Kleiman's theorem applied to the morphisms $i, \underline{\nu}$ implies that there exists a dense open set $V_1 \subset \underline{G}$ such that the inverse image of $\underline{\Gamma}^\sigma$ in $(M^*)^c$ for any $\sigma \in V_1$, is empty. Therefore generally, the intersection is wholly supported in M^*.

Now we prove the intersection consists of finite number of reduced points.

Let $Y := \mathrm{Sing}\,\underline{\Gamma}$, the set of singular points, it is indeed a variety by Theorem 5.3 of Hartshorne [7]. Now we apply Kleiman's theorem to the inclusion

$$Y \hookrightarrow X^n$$

and

$$\underline{\mathrm{ev}} : M^* \to X^n$$

we obtain a dense open set $V_2 \subset \underline{G}$ such that either □

Corollary 4.4 *The intersection of $\underline{\mathrm{ev}}^{-1}(\underline{\Gamma})$ with any boundary divisor in $\overline{M}_{0,n}(\mathbb{P}^r, d)$ is transversal.*

Proof Any such intersection should still be a finite number of reduced points. □

4.2 Counting Maps

In the proof of Kontsevich's formula, we will be counting the number of maps (points) in $\overline{M}_{0,n}(\mathbb{P}^r, d)$ that meet prescribed conditions. We prove here in this section the equivalence of counting these with counting the number of rational curves of degree d passing through $3d - 1$ points.

Lemma 4.5 *Suppose $n \geq 2$, and $1 \leq i, j \leq n$ are distinct. Let*

$$Q_{ij} := \{\mu \in M_{0,n}(\mathbb{P}^r, d) : \mu(p_i) = \mu(p_j)\}$$

be the locus of maps whose two marked points $p_i \neq p_j$ have common image in $\mathbb{P}^r$. Then the codimension of Q_{ij} in $M := M_{0,n}(\mathbb{P}^r, d)$ is equal to r.

Note in particular that we are considering $M_{0,n}(\mathbb{P}^r, d)$ and not $\overline{M}_{0,n}(\mathbb{P}^r, d)$. As Proposition 4.3 suggests, our constraints will be supported wholly on the locus with smooth source curve.

Proof We claim that it suffices to assume that $n \geq 3$. Indeed, consider the commutative diagram

$$\begin{array}{ccc} M_{0,n+1}(\mathbb{P}^r, d) & \xrightarrow{\hat{\nu}_i} & \mathbb{P}^r \\ \varepsilon \downarrow & \nearrow_{\nu_i} & \\ M_{0,n}(\mathbb{P}^r, d) & & \end{array}$$

where ε is the forgetful map of the $n+1$-th marked point, whereas $\hat{\nu}_i$ and ν_i are the maps of evaluation at p_i for the respective spaces. From commutativity we also get that $\hat{\nu}_i^{-1}(Q_{ij}) = \varepsilon^{-1}\nu_i^{-1}(Q_{ij})$ Recall from Definition 3.1 that the map μ is defined by $r+1$ degree d forms. Denote

$$y_k = \sum_{j=0}^{d} a_{kj} x_0^j x_1^{d-j}$$

be the k-th form. Since we assume $n \geq 3$ we can assume the marked points have homogeneous coordinates $p_i = [0 : 1]$ and $p_j = [1 : 0]$ (there is uniqe automorphism of $\mathbb{P}^1$ that sends the marked points to 1 and 0 respectively). Now the condition $\mu(p_i) = \mu(p_j)$ is equivalent to

$$(a_{00}, a_{10}, \dots, a_{r,0}) = \lambda(a_{0d}, a_{1d}, \dots, a_{rd})$$

for some $\lambda \in \mathbb{C}^*$. This is r independent conditions in the a_{1j}. □

Lemma 4.6 *Let $r \geq 2$. For generic choices of $\Gamma_1, \dots, \Gamma_n \subset \mathbb{P}^r$, with codimensions adding up to* $\dim \overline{M}_{0,n}(\mathbb{P}^r, d)$*, we have*

$$\mu^{-1}\mu(p_i) = \{p_i\}, i = 1, \dots, n \quad (\textit{with multiplicity } 1)$$

for every map μ in the intersection $\underline{\nu}^{-1}(\underline{\Gamma})$.

For the proof of this lemma, we need to consider the dense open set $M_{0,n}^{\circ}(\mathbb{P}^r, d) \subset \overline{M}_{0,n}(\mathbb{P}^r, d)$ consisting of maps which have smooth source and are immersions, i.e. that the tangent map is injecive when we consider the map as a pointed map $\mathbb{P}^1 \to \mathbb{P}^r$. To establish that this is indeed a dense open subset of maps, we need the following lemma

Lemma 4.7 *Let $r \geq 2$. The locus $W^{\circ}(r, d) \subset W(r, d)$ consisting of immersions is open, thus is dense in the irreducible variety $W(r, d)$.*

Proof Consider the closed subset in $W(r,d) \times \mathbb{P}^1$:

$$\Sigma := \{(\mu, x) \in W(r,d) \times \mathbb{P}^1 \mid D\mu_x = 0\}$$

Then $W(r,d) \setminus W^\circ(r,d)$ is the image of of the projection $\Sigma \to W(r,d)$ which is closed since the projection is a closed map. □

Now we claim that by this lemma that we have that $M^\circ_{0,n}(\mathbb{P}^r, d) \subset M_{0,n}(\mathbb{P}^r, d)$ is an open dense subset, and thus also an open dense subset of the compactification. Indeed, by construction we know that $M_{0,n}(\mathbb{P}^r, d) = M_{0,n} \times W(r,d)$ so in fact we have

$$M^\circ_{0,n}(\mathbb{P}^r, d) = M_{0,n} \times W^\circ(r,d)$$

and the claim follows immediately from the lemma.

Proof (PROOF OF LEMMA 4.6) By Proposition 4.3, the intersection $\underline{\nu}^{-1}(\underline{\Gamma})$ consists of a finite number of reduced points, supported on any chosen dense subset of $\overline{M}(\mathbb{P}^r, d)$. We choose this dense open subset to be the dense open subset $M^\circ_{0,n}(\mathbb{P}^r, d) \subset \overline{M}_{0,n}(\mathbb{P}^r, d)$. Now notice that a map $\mathbb{P}^1 \to \mathbb{P}^r$ having injective tangent map is equivalent to the map having no ramifications, this immediately implies that $\mu^{-1}\mu(p_i)$ is reduced for each $i = 1, \ldots, n$ for any μ in the intersection $\underline{\nu}^{-1}(\underline{\Gamma})$ which is supported in the locus $M^\circ_{0,n}(\mathbb{P}^r, d)$. Now for each i, define $J_i \subset M^\circ_{0,n}(\mathbb{P}^r, d)$ to be the locus of maps μ for which $\mu^{-1}\mu(p_i)$ contains at least one point distinct from p_i. We aim to show that this locus has positive codimension inside $M^\circ_{0,n}(\mathbb{P}^r, d)$, which means we can perform another transversality argument. To that end, we consider the space $M^\circ_{0,n+1}(\mathbb{P}^r, d)$ with one extra mark which we denote p_0, and consider the forgetful map

$$\varepsilon : M^\circ_{0,n+1}(\mathbb{P}^r, d) \to M^\circ_{0,n}(\mathbb{P}^r, d)$$

that forgets p_0. We claim that the image of $Q_{i,0}$ (as defined in Lemma 4.5) under this forgetful map is exactly $J_i \subset M^\circ_{0,n}(\mathbb{P}^r, d)$. That $\operatorname{Im} Q_{i,0} \subset J_i$ is trivially a result by the definitions of these sets involved. We show the other inclusion. For any $\mu \in J_i$, we know that there exists some point in the source of μ distinct from p_i that has the image of p_i. Putting the extra mark p_0 on this point we get an $(n+1)$-pointed map that is in $Q_{i,0}$ and whose image under ε is μ. This establishes that $J_i \subset \operatorname{Im} Q_{i,0}$.

Now by Lemma 4.5, $Q_{i.0}$ has codimension $r \geq 2$, we can conclude that J_i must have codimension at least $r - 1 \geq 1$, as desired. □

Now let us recall the object of interest in Kontsevich's formula.

Definition 4.8 Let N_d denote the number of rational curves of degree d passing through $3d - 1$ general points in $\mathbb{P}^2$.

Then the previous Lemma implies that counting maps is equivalent to counting curves, more precisely:

Corollary 4.9 *If $\Gamma_1, \ldots, \Gamma_{3d-1}$ are general points in $\mathbb{P}^2$, then the number of stable $(3d-1)$-pointed maps $\mu : C \to \mathbb{P}^r$ (with marked points $\{p_1, \ldots, p_{3d-1}\}$) such that $p_i \mapsto \Gamma_i$ is equal to the number N_d of rational curves through those points.*

Proof By Lemma 4.6, each map passes only once through each point, so there is precisely one possibility for the position of each marked point. Thus the number of stable maps is also the number of rational curves passing through the points. □

4.3 Bézout's Theorem

Here we restate the well known Bézout's Theorem which we will use in the proof of Kontsevich's formula. The proof can be found in many elementary textbooks on algebraic geometry, for example in Hartshorne [7].

Theorem 4.10 (Bézout's Theorem) *Let Y, Z be distinct curves in $\mathbb{P}^2$, having degrees d, e. Then $Y \cap Z$ consists of $d \cdot e$ points, counting multiplicities.*

Proof See Hartshorne [7] Corollary I.7.8. □

4.4 Kontsevich's Formula for Rational Plane Curves

We seek to find a formula that gives the number N_d, for any d.

Why $3d-1$ points? Consider the space $\mathbb{P}^{d(d+3)/2}$ of homogeneous polynomials of degree d. The space of all irreducible rational curves of degree d constitutes a subvariety $V_0^d \subset \mathbb{P}^{d(d+3)/2}$ of dimension given by the genus-degree formula:

$$g = \frac{(d-1)(d-2)}{2} - \delta$$

where g is the genus, and δ is the number of nodes. We are only considering genus 0 curves, thus we have the equation

$$\frac{(d-1)(d-2)}{2} = \delta$$

As such, to give a rational curve of degree d we must impose $(d-1)(d-2)/2$ nodes. Further, imposing each node is a condition of codimension 1 (i.e. the condition corresponds to a hypersurface), we actually have that

$$\dim V_0^d = \frac{d(d+3)}{2} - \frac{(d-1)(d-2)}{2} = 3d - 1$$

this means that to get a finite number of irreducible rational curves of degree d we must impose $3d-1$ conditions, e.g. the condition of passing through $3d-1$ general points.

Lemma 4.11 $N_1 = 1$.

Proof This is obvious, there is only one unique line that passes through any given two distinct points. $\square$

Theorem 4.12 (Kontsevich's Formula) *The following recursive relation holds:*

$$\begin{aligned} N_d + \sum_{\substack{d_A+d_B=d \\ d_A\geq 1, d_B\geq 1}} \binom{3d-4}{3d_A-1} d_A^2 N_{d_A} \cdot N_{d_B} \cdot d_A d_B \\ = \sum_{\substack{d_A+d_B=d \\ d_A\geq 1, d_B\geq 1}} \binom{3d-4}{3d_A-2} d_A N_{d_A} \cdot d_B N_{d_B} \cdot d_A d_B \end{aligned}$$

Proof Set $n := 3d$. We will work in the moduli space $\overline{M}_{0,n}(\mathbb{P}^2, d)$, a variety of dimension $6d-1$ (by Proposition 3.29). We use symbols $m_1, m_2, p_1, \ldots, p_{n-2}$ to indicate the marks. Take two lines L_1, L_2 in $\mathbb{P}^2$ and $n-2$ points $Q_1, \ldots, Q_4$, all in general position.

Let $Y \subset \overline{M}_{0,n}(\mathbb{P}^2, d)$ be the subset consisting of maps

$$(C; m_1, m_2, p_1, \ldots, p_{n-2}; \mu) \text{ such that } \begin{cases} \mu(m_1) & \in L_1 \\ \mu(m_2) & \in L_2 \\ \mu(p_i) & = Q_i, \quad i = 1, \ldots, n-2 \end{cases}$$

In fact, Y is a subvariety since it is the intersection of the following inverse images under the evaluation maps:

$$Y = \mathrm{ev}_{m_1}^{-1}(L_1) \cap \mathrm{ev}_{m_2}^{-1}(L_2) \cap \mathrm{ev}_{p_1}^{-1}(Q_1) \cap \cdots \cap \mathrm{ev}_{m_{n-2}}^{-1}(Q_{n-2})$$

By flatness of the evaluation maps, the inverse image of a line is of codimension 1, and the inverse image of a point is of codimension 2, so the total codimension of the intersection is $2 + 2(n-2) = 6d - 2$, i.e. Y is a curve. We will now consider the intersection of Y with boundary divisors. Such an intersection will be a subvariety of codimension $2 + 2(n-2) + 1 = 6d - 1$ equal to the dimension of $\overline{M}_{0,n}(\mathbb{P}^2, 2)$ because boundary divisors have codimension 1 and the intersection is a finite number of reduced points. Furthermore, we know that the intersection takes place in any chosen open dense subset. We will choose the smooth locus $M_{0,n}^*(\mathbb{P}^2, 2) \subset \overline{M}_{0,n}(\mathbb{P}^2, 2)$. The generality of the points and lines imply (by Proposition 4.3 and Corollary 4.4) that Y intersects each boundary divisor transversally in our chosen dense open set $M_{0,n}^*(\mathbb{P}^2, 2)$.

Consider the forgetful map $\overline{M}_{0,n}(\mathbb{P}^2, d) \to \overline{M}_{0,\{m_1,m_2,p_1,p_2\}} = \overline{M}_{0,4}(\mathbb{P}^2, d)$ which forgets the map μ and the marks $p_3, \ldots, p_{n-2}$ as discussed in Theorem 3.5 to obtain the fundamental relation. From the fundamental relation we obtain

$$Y \cap D^{-1}(m_1, m_2 | p_1, p_2) \equiv Y \cap D^{-1}(m_1, p_1 | m_2, p_2) \tag{4.1}$$

First consider the left hand side. Recalling how we define the boundary divisor

$$D^{-1}(m_1, m_2|p_1, p_2) = \sum D(A, B; d_A, d_B)$$

where the sum runs over all d-weighted partitions of the markings

$$\{m_1, m_2, p_1, \dots, p_{n-2}\}$$

Each term in the sum is an irreducible component of the divisor corresponding to a choice of distributing marks and degrees. We now investigate the intersection of Y with each of the irreducible boundary divisors. We will show that by by counting the maps in the intersections we will obtain the claimed equality of Kontsevich's formula. Throughout, we will use Corollary 4.9 to to translate between counting maps and counting curves in $\mathbb{P}^2$.

First consider the irreducible boundary divisors where the partial degrees is distributed as $d_A = 0$, $d_B = d$, i.e. all the degrees are on the B-twig. This means that the A-twig C_A must be mapped to a point $z \in \mathbb{P}^2$ (under μ), since the restriction of μ to C_A must be of degree d_A. Recall that the marked point m_1 maps to L_1 and the marked point m_2 maps to L_2, and since the boundary divisor $D^{-1}(m_1, m_2|p_1, p_2)$ must have these two points on the A-twig, we must have $\{z\} = L_1 \cap L_2$. Suppose there were more than these two marked points on the A-twig, then these marked points would also be mapped to z, but that would contradict the assumption that the lines and points are in general position. Thus in this case Y has empty intersection with all the cases where there are more than just m_1 and m_2 on the A-twig. So when we only have m_1 and m_2 on the A-twig, we must map the B-twig to a degree d curve in $\mathbb{P}^r$ (since $d_B = d$). Once this conic is fixed there are no more choices left for the marked points since the node $C_A \cap C_B$ maps to z, and the other marked points map to the Q_i's. Now notice that the number of ways to draw a degree d curve that passes through $n - 2 + 1 = 3d - 1$ points ($p_1, \dots, p_{n-2}$ plus the node $C_A \cap C_B$) is exactly N_d, by Corollary 4.9 this is also the number of maps that map to a curve that passes through these points. This gives the first term on the left hand side of the desired equality.

Similarly, let us consider the cases where $d_B = 0$. This means that C_B maps to a point. However, since there will always be more than two marked points (excluding m_1 and m_2 since they have to be on the A-twig) mapped to the same point, this contradicts the assumption of general position of the Q_i's. Therefore there is no contribution from these cases.

Now we consider the cases where the partial degrees d_A and d_B are positive. We claim that the only distributions of the additional marks (i.e. marks other than m_1, m_2, p_1, p_2) giving contribution is when $3d_A - 1$ additional marks fall on the A-twig. This is because C_A must be mapped to a curve of degree d_A, and if more than $3d_A - 1$ additional points are placed on C_A, that means together with m_1 and m_2 a total of $3d_A + 1$ points are mapped to a degree d_A curve which contradicts the generacity assumption. The same argument also implies there can be no more than $3d_B - 3$ additional points placed on C_B. Therefore we have $\binom{3d-4}{3d_A-1}$ ways to distribute the additional marks.

Now there are N_{d_A} possible curves for the image of C_A and N_{d_B} possible curves for the image of C_B. Once these are determined, the p_i's are determined.

Now it remains to consider how many ways to send the points m_1 and m_2. We know that m_1 has to be sent to a point of intersectoin between $\mu(C_A)$ and L_1, by Bézout's Theorem there are d_A such possibilities. The same is true for m_2. This explains the factor of d_A^2.

The intersection point $x \in C_A \cap C_B$ must be sent to a point of intersection between $\mu(C_A)$ and $\mu(C_B)$. By Bézout's Theorem again, there are $d_A d_B$ such possibilities.

At this point we have completed the examination of the left-hand side of (4.1) and showed that it coincides with the left-hand side of Kontsevich's formula.

Now for the right-hand side. For the right-hand side of (4.1), we must consider boundary divisors with m_1, p_1 on the A-twig, and m_2, p_2 on the B-twig,

If either d_A or d_B is zero, then $Q_1 \in L_1$ or $Q_2 \in L_2$, respectively. This is a contradiction to the generality of the Q_i's and L_i's.

For the other choices of d_A and d_B, using the same genericity argument as before, the only contribution comes from the cases when $3d_A - 2$ additional points are placed on the A-twig, and there are $\binom{3d-4}{3d_A-2}$ such possibilities. For each of these, the curves $\mu(C_A)$ and $\mu(C_B)$ can be chosen in N_{d_A} and N_{d_B} ways, respectively. Now the marked point m_1 must be mapped to $\mu(C_A) \cap L_1$. By Bézout's there are exactly d_A choices. Similarly we have d_B choices to map the marked point m_2.

Finally we have to consider where to send the point $x \in C_A \cap C_B$. We can choose any one of the $d_A d_B$ points of intersection of $\mu(C_A)$ and $\mu(C_B)$.

This completes the analysis of the right-hand side of (4.1), which shows it coincides with the right-hand side of Kontsevich's formula.

Thus we have the desired result. □

Chapter 5

Gromov-Witten Invariants

As mentioned earlier, the proof of the main result we gave was not the original proof of Kontsevich and Manin. This should not come as a surprise since the line of argument we made begins with the formula as a given, and all we did was verify the terms in the equation. It is therefore difficult to believe that this is how it was originally conceived. Indeed, the technique used by Kontsevich and Manin was a much greater machinery, that of *Gromov-Witten invariants*, which roughly speaking, counts the number of maps in the moduli space of stable maps meeting prescribed incidence conditions. Of course, this already sounds exactly like what Kontsevich's formula counts, some number of maps of curves that meet conditions of meeting a number of points.

Gromov-Witten invariants were originally inspired by ideas in theoretical physics, and were defined in the context of symplectic geometry. It is a remarkable fact that physics has led to great developments in pure mathematics; and has continued to do so in the past few decades. For more detailed information on Gromov-Witten invariants related to theoretical physics, see [2] and [8].

In this chapter, we will introduce Gromov-Witten invariants for the genus-0 case. That is, we (still) consider the moduli space of stable maps with source as a genus-0 nodal curve; however, we allow ourselves to consider maps to an arbitrary smooth homogeneous variety X. The existence of the moduli spaces with such a target X (instead of $\mathbb{P}^r$ as we have been working with) is in fact presented in F-P, so in fact when we presented these statements in the previous chapters, their original statements in the source were of this level of generality. As a result, we will not discuss the existence of any moduli spaces that will appear in the following two chapters. In fact, F-P constructs these moduli spaces of maps for arbitrary genus source curve, but we will not need this.

Note that Gromov-Witten invariants can be defined in much greater generality, what we present here is arguably the simplest flavor. Throughout, we will work with cohomology with $\mathbb{Q}$ coefficients.

5.1 Definition

We fix a non-singular projective variety X that is a homogeneous space. Thus it satisfies the conditions given by Example 19.1.11. of [5], that there is an isomorphism between the Chow groups and singular homology groups, with a doubling of degree:

$$A_d(X) = H_{2d}(X)$$

When X is of dimension r, the intersection ring $A^*(X)$ is defined by setting $A^d(X) := A_{r-d}(X)$. Then by Poincaré duality isomorphism we can identify A^*X with A_*X:

$$\begin{aligned} A^*(X) &\to A_*(X) \\ \gamma &\mapsto \gamma \cap [X] \end{aligned}$$

so we also have equality of operational classes (classes in A^*) and cohomology classes:

$$A^d(X) = H^{2d}(X)$$

We will only be concerned with (co)homology classes of even degree, so we will seldom distinguish between homology classes and algebraic cycle classes, and between cohomology classes and operational classes.

We will however, work with cohomology with $\mathbb{Q}$ coefficients.

We will consider the more general moduli space $\overline{M}_{0,n}(X, \beta)$ which parametrizes stable pointed maps $(C, p_1, \ldots, p_n, \mu)$ where C is a genus-0 nodal curve and X is a non-singular projective variety, $\beta \in A_1(X) = H_2(X)$ and $\mu_*([C]) = \beta$. It is immediately clear that if $\beta \neq 0$ the moduli space is empty unless β is the class of a curve. In particular, when $X = \mathbb{P}^r$, then since $0 \neq \beta \in A_1(X)$ is determined by the degree of a line, we write $\overline{M}_{0,n}(\mathbb{P}^r, d)$ instead of $\overline{M}_{0,n}(\mathbb{P}^r, d[\text{line}])$. Thus we recover the moduli space that is more familiar to us.

Recall that our proof of Kontsevich's formula involved counting the finite number of points in the set $\underline{\text{ev}}^{-1}(\underline{\Gamma})$, where

$$\underline{\Gamma} = \Gamma_1 \times \cdots \times \Gamma_n$$

is the product of subvarieties in $\mathbb{P}^r$ whose codimensions add up to the dimensions of $\overline{M}_{0,n}(\mathbb{P}^r, d)$. This is equivalent to computing the degree of the algebraic cycle class $[\underline{\text{ev}}^{-1}(\underline{\Gamma})]$:

$$\deg([\underline{\text{ev}}^{-1}(\underline{\Gamma})])$$

since by proposition 4.3, $\underline{\mathrm{ev}}^{-1}(\underline{\Gamma})$ consists of a finite number of reduced points, each point contributes one value to the sum.

Now let us reformulate this degree count in terms of homology and cohomology classes. Let $\gamma_i \in H^*(\mathbb{P}^r)$ be the cohomology classes corresponding to the algebraic cycle class $[\Gamma_i] \in A_*(\mathbb{P}^r)$ (via Poincaré duality). Then consider the cohomology class

$$\mathrm{ev}_1^*(\gamma_1) \cup \cdots \cup \mathrm{ev}_n^*(\gamma_n) \in H^*(\overline{M}_{0,n}(\mathbb{P}^r, d))$$

Then using the correspondence between algebraic cycle classes and (co)homology classes we can rewrite

$$\deg([\underline{\mathrm{ev}}^{-1}(\underline{\Gamma})]) = \int_{\overline{M}_{0,n}(\mathbb{P}^r,d)} \mathrm{ev}_1^*(\gamma_1) \cup \cdots \cup \mathrm{ev}_n^*(\gamma_n)$$

where on the right-hand side we are taking the value of the cohomology class $\mathrm{ev}_1^*(\gamma_1) \cup \cdots \cup \mathrm{ev}_n^*(\gamma_n)$ on the fundamental homology class $[\overline{M}_{0,n}(\mathbb{P}^r, d)]$.

This will motivate the following definition:

Definition 5.1 (Gromov-Witten invariant) Let $\gamma_1, \ldots, \gamma_n \in A^*(X)$ be arbitrary classes. We define the *Gromov-Witten invariant* to be

$$I_\beta(\gamma_1 \cdots \gamma_n) = \int_{\overline{M}_{0,n}(X,\beta)} \mathrm{ev}_1^*(\gamma_1) \cup \cdots \cup \mathrm{ev}_n^*(\gamma_n)$$

It follows directly from the definition that $I_\beta(\gamma_1 \cdots \gamma_n)$ is invariant under permutation of the γ_i's, which is reflected by the notation $\gamma_1 \cdots \gamma_n$ as a product.

5.2 Properties

In Chapter 3, we cited important results from [6] on the existence of $\overline{M}_{0,n}(\mathbb{P}^r, d)$. In fact, [6] presents results that are more general and concern the existence of $\overline{M}_{g,n}(X, \beta)$ as a coarse moduli space, and that further $\overline{M}_{0,n}(X, \beta)$ is a projective variety. In particular, the dimension of this variety is given:

Theorem 5.2 (F-P [6] Theorem 2) *Let X be a projective, non-singular, convex variety. Then $\overline{M}_{0,n}(X, \beta)$ is a normal projective variety of pure dimension*

$$\dim(X) + \int_\beta c_1(T_X) + n - 3$$

where $c_1(T_X)$ is the first Chern class of the tangent bundle T_X of X.

This is relevant to us since in the definition of the Gromov-Witten invariant, unless the cohomology class

$$\mathrm{ev}_1^*(\gamma_1) \cup \cdots \cup \mathrm{ev}_n^*(\gamma_n)$$

has component of top degree in $H^*(\overline{M}_{0,n}(X,\beta))$, the value is zero. Equivalently, using Poincaré duality to relate the dimension and codimension of (co)homology classes, the Gromov-Witten invariant vanishes unless

$$\sum \mathrm{codim}(\gamma_i) = \dim(\overline{M}_{0,n}(X,\beta)) = \dim X + \int_\beta c_1(T_X) + n - 3$$

The following proposition illustrates that this definition is natural in the context of our earlier work, that is, to find a formula for N_d. In fact, it is more natural than what we have done in the proof of Kontsevich's formula in the following sense. If we are imposing $3d-1$ incidence conditions, then we should have $3d-1$ subvarieties Γ_i of dimension 0 (i.e. points) representing those conditions in $\mathbb{P}^2$; while having the same number of marked points on the source curve, each corresponding to an incidence condition. Thus one naturally considers the moduli space $\overline{M}_{0,3d-1}(\mathbb{P}^2, d)$, and look at the pullbacks of the evaluation maps from the subvarieties to see how many maps meet all those conditions. Of course, genericity in the position of the subvarieties must be accounted for so we should phrase them in terms of algebraic cycle classes or (co)homology classes.

Proposition 5.3 *Let $\gamma_1, \cdots, \gamma_n \in A^*(\mathbb{P}^r)$ be classes of codimension at least 2, with $\sum \mathrm{codim}\, \gamma_i = \dim \overline{M}_{0,n}(\mathbb{P}^r, d) = rd + r + d + n - 3$. Then for subvarieties $\Gamma_1, \ldots, \Gamma_n \subset \mathbb{P}^r$ in general position with*

$$[\Gamma_i] = \gamma_i \cap [\mathbb{P}^r]$$

the Gromov-Witten invariant $I_d(\gamma_1 \cdots \gamma_n)$ is the number of rational curves of degree d that are incident to all the subvarieties $\Gamma_1, \cdots, \Gamma_n$.

However, recall from the proof of Kontsevich's formula, that to obtain a formula for N_d, we considered instead the moduli space $\overline{M}_{3d}$, and considered a mixture of subvarieties of dimension 1 and 0. Subsequently we performed a series of combinatorial analysis on the boundary to obtain the desired recursive relation. The appearance of N_d is somewhat unexpected and seem to have obscure origins and not at all clear until its appearance that the approach is the correct one. It is in this sense that we mean that the above formulation of N_d in terms of Gromov-Witten invariants is more natural.

We end with some properties of the Gromov-Witten invariant which we will need later for studying quantum cohomology.

Suppose $\beta = 0$. That is, $\overline{M}_{0,n}(X,\beta)$ is the moduli space of constant maps $(C, p_1, \ldots, p_n) \to X$. In which case $\overline{M}_{0,n}(X,\beta) = \overline{M}_{0,n} \times X$ where the second factor is the value of the map. Then in particular the evaluation maps ev_i are all identical to the projection onto the second factor

$$p\colon \overline{M}_{0,n} \times X \to X$$

Lemma 5.4 *Suppose $\beta = 0$. Then*

$$I_\beta(\gamma_1 \cdots \gamma_n) = \int_{p_*[\overline{M}_{0,n} \times X]} \gamma_1 \cup \cdots \cup \gamma_n$$

Proof We use the identity

$$\mathrm{ev}_1 * (\gamma_1) \cup \cdots \cup \mathrm{ev}_n^*(\gamma_n) = p^*(\gamma_1 \cup \cdots \cup \gamma_n)$$

to obtain

$$\begin{aligned} I_\beta(\gamma_1 \cdots \gamma_n) &= \int_{\overline{M}_{0,n}(X,\beta)} \mathrm{ev}_1^*(\gamma_1) \cup \cdots \cup \mathrm{ev}_n^*(\gamma_n) \\ &= \int_{\overline{M}_{0,n} \times X} p^*(\gamma_1 \cup \cdots \cup \gamma_n) \\ &= \int_{p_*[\overline{M}_{0,n} \times X]} \gamma_1 \cup \cdots \cup \gamma_n \end{aligned}$$ □

One particular consequence of this lemma is the following:

Corollary 5.5 *If $\beta = 0$, then $I_\beta(\gamma_1 \cdots \gamma_n)$ is non-zero if and only if $n = 3$.*

Proof We use the equality of Lemma 5.4 throughout.

In the case that $0 \leq n \leq 2$, the space $\overline{M}_{0,n}$ is empty, so the Gormov-Witten invariant is trivially zero. Assume that $n \geq 3$. For $p_*[\overline{M}_{0,n} \times X]$ to be non-zero, it is equivalent to having

$$\dim(p(\overline{M}_{0,n} \times X)) = \dim(X) = \dim(\overline{M}_{0,n} \times X)$$

(see [5]) §1.4 for the definition of push-forward of cycles). Now for $n > 3$, the space $\overline{M}_{0,n}$ has positive dimension, therefore the above equality cannot hold. It is only when $n = 3$, i.e. $\overline{M}_{0,3}$ is a point, that the equality can hold. This implies the assertion. □

Chapter 6

Quantum Cohomology

We will now introduce a new binary operation on the cohomology classes of X, called the *quantum cup product*, which is defined using generating functions for Gromov-Witten invariants, which are called *Gromov-Witten potentials*. We will show that this binary operation is commutative and associative. In particular, the associative property will be equivalent to the recursive formula for N_d.

In preparation for the definition of the Gromov-Witten potential, we quickly recall the notion of generating functions.

6.1 Generating Functions

A generating function is a series which one uses to package a sequence of numbers. More precisely, suppose $\{N_k\}_{k=0}^{\infty}$ is a sequence of numbers, then we can form the following power series called the *generating function* (of the sequence):

$$F(x) := \sum_{k=0}^{\infty} \frac{x^k}{k!} N_k$$

Obviously, there are many other ways one can define a series taking the entries of the sequence as coefficients, and the above example is more precisely the so-called *exponential generating function*. But since we will not have the need for other types, we will just call it the generating function associated to the sequence.

There are many reasons for defining generating functions, and they more or less come down to manipulations of the series to obtain some relation or property of the sequence itself. In particular for our purposes, we will often consider the formal derivative:

$$F_x := \frac{\mathrm{d}}{\mathrm{d}x} F$$

which is the generating function for the sequence $\{N_{k+1}\}_{k=0}^{\infty}$.

We can also multiply generating functions to get another generating function:

Lemma 6.1 (Product rule for generating functions) *Suppose we have two generating functions*

$$F(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} f_k \quad \textit{and} \quad G(x) = \frac{x^k}{k!} g_k$$

then their product is the generating function for the numbers

$$h_k = \sum_{i=0}^{k} \binom{k}{i} f_i g_{k-1}$$

Proof

$$\begin{aligned} F \cdot G(x) &= \left(\sum_{k=0}^{\infty} \frac{x^k}{k!} f_k \right) \left(\sum_{j=0}^{\infty} \frac{x^k}{j!} g_j \right) \\ &= \sum_{k,j} \frac{x^{k+j}}{k! j!} f_k g_j \\ &= \sum_{m=0}^{\infty} x^m \left(\sum_{m+j=k} \frac{f_k g_j}{k! j!} \right) \\ &= \sum_{m=0}^{\infty} \left(\sum_{k=0} \binom{m}{k} f_k g_{m-k} \right) \end{aligned}$$

□

6.2 Gromov-Witten Potential and the Quantum Cup Product

We fix a basis

$$\{T_0, T_1, \ldots, T_m\}$$

for the cohomology groups of X. In particular we let $T_0 = 1 \in A^0(X)$, and $T_1, \ldots, T_p$ a basis for $A^1(X)$, and $T_{p+1}, \ldots, T_m$ a basis for the other cohomology groups.

So the possible input classes

$$T_0^{n_0} \cdots T_m^{n_m}$$

for the Gromov-Witten invariant are parametrized by the index variables $n_0, \ldots, n_m$.

In addition, for $0 \leq i, j \leq m$, define

$$g_{ij} = \int_X T_i \cup T_j$$

We also define the numbers g^{ij} to be the entries of the matrix $(g^{ij}) := (g_{ij})^{-1}$.

For notational simplicity, let us collect the Gromov-Witten invariants with input class $\gamma_1, \ldots, \gamma_n$, into the *collected Gromov-Witten invariants*, which is a formal sum over all possible choices of β:

$$I(\gamma_1, \ldots, \gamma_n) = \sum_\beta I_\beta(\gamma_1, \ldots, \gamma_n)$$

The Gromov-Witten potential will be the generating function for the collected Gromov-Witten invariants. We introduce formal variables

$$\boldsymbol{y} = (y_0, \ldots, y_m)$$

for the generating function. Then,

Definition 6.2 (Gromov-Witten potential)

$$\Phi(y_0, \ldots, y_m) = \sum_{n_0+\cdots+n_m \geq 3} \frac{y_0^{n_0}}{n_0!} \cdots \frac{y_m^{n_m}}{n_m!} I(T_0^{n_0} \cdots T_m^{n_m}) \tag{6.1}$$

or, using multi index notation, where

$$\boldsymbol{\nu} = (n_0, \ldots, n_m)$$

and understanding that we only consider n_i's such that $n_0 + \cdots + n_m \geq 3$. And put

$$\boldsymbol{y}^{\boldsymbol{\nu}} = y_0^{n_0} \cdots y_{n_m}^{n_m} \quad \text{and} \quad \boldsymbol{\nu}! = n_0! \cdots n_m! \quad \text{and} \quad \boldsymbol{h}^{\boldsymbol{\nu}} = T_0^{a_0} \cdots T_m^{a_m}$$

Then we can write the Gromov-Witten potential as

$$\Phi(\boldsymbol{y}) = \sum_{\boldsymbol{\nu}} \frac{\boldsymbol{y}^{\boldsymbol{\nu}}}{\boldsymbol{\nu}!} I(\boldsymbol{h}^{\boldsymbol{\nu}})$$

Note that the Gromov-Witten potential is a formal power series with $\mathbb{Q}$ coefficients, since we consider cohomology with $\mathbb{Q}$ coefficients.

Now define Φ_{ijk} to be the partial derivative:

$$\Phi_{ijk} = \frac{\partial^3 \Phi}{\partial y_i \partial y_j \partial y_k}, \quad 0 \leq i, j, k \leq m$$

The following formula will be useful. But first, let us introduce yet another formal variable to make things look nicer. Let $\gamma = \sum y_i T_i$.

Lemma 6.3

$$\Phi_{ijk} = \sum_{\boldsymbol{\nu}} \frac{\boldsymbol{y}^{\boldsymbol{\nu}}}{\boldsymbol{\nu}!} I(\boldsymbol{h}^{\boldsymbol{\nu}} \cdot T_i \cdot T_j \cdot T_k) = \sum_{n=0}$$

Proof Let us consider first the partial derivative with respect to one variable.

$$\begin{aligned}
\frac{\partial \Phi(y_0, \ldots, y_m)}{\partial y_i} &= \sum_{\boldsymbol{\nu}} \frac{a_1 y_0^{n_0} \cdots y_i^{a_i - 1} \cdots y_r^{a_r}}{a_0! \cdots a_r!} I(T_0^{n_0} \cdots T_m^{n_m}) \\
&= \sum_{\boldsymbol{\nu}} \frac{y_0^{n_0} \cdots y_i^{a_i - 1} \cdots y_r^{a_r}}{a_0! \cdots (a_i - 1)! \cdots a_r!} I(T_0^{n_0} \cdots T_m^{n_m}) \\
&= \sum_{\boldsymbol{\nu}} \frac{y_0^{n_0}}{n_0!} \cdots \frac{y_m^{n_m}}{n_m!} I(T_0^{n_0} \cdots T_m^{n_m} \cdot T_i) \\
&= \sum_{\boldsymbol{\nu}} \frac{\boldsymbol{y}^{\boldsymbol{\nu}}}{\boldsymbol{\nu}!} I(\boldsymbol{h}^{\boldsymbol{\nu}} \cdot T_i)
\end{aligned}$$

Repeating the same procedure twice, we obtain the desired result. □

Notice that this identity says that Φ_{ijk} is the generating function for the sequence of collected Gromov-Witten numbers $I(\boldsymbol{h}^{\boldsymbol{\nu}} \cdot T_i \cdot T_j \cdot T_k)$. That is, considered as a sequence indexed by all possible $\boldsymbol{\nu}$. This will be important later.

Now we finally arrive at a definition of the *quantum cup product* on the T_i's:

Definition 6.4 (Quantum cup product)

$$T_i * T_j = \sum_{e,f} \Phi_{ije} g^{ef} T_f$$

it is evident from this definition that the quantum cup product is commutative, since the partial derivatives are symmetric in the subscripts.

Lemma 6.5 *The quantum cup product is commutative.*

Proposition 6.6 *A unit for the quantum cup product is $T_0 = 1$.*

Proof First we write out the value of Φ_{0jk} using Lemma 6.3 and Lemma 5.4:

$$\begin{aligned}
\Phi_{0jk} &= \sum_{\boldsymbol{\nu}} \frac{\boldsymbol{y}^{\boldsymbol{\nu}}}{\boldsymbol{\nu}!} I(\boldsymbol{h}^{\boldsymbol{\nu}} \cdot T_0 \cdot T_j \cdot T_k) \\
&= \sum_{\boldsymbol{\nu}} \frac{\boldsymbol{y}^{\boldsymbol{\nu}}}{\boldsymbol{\nu}!} \sum_{\beta} I_\beta(\boldsymbol{h}^{\boldsymbol{\nu}} \cdot T_0 \cdot T_j \cdot T_k) \\
&= I_0(T_0 \cdot T_j \cdot T_k) \\
&= \int_X T_j \cup T_k \\
&= g_{jk}
\end{aligned}$$

Crucial to the above derivation is the fact that when the fundamental class T_0 is present, the only non-zero Gromov-Witten invariants are degree zero and with three marks.

Then

$$\begin{aligned}\Phi_0 * T_j &= \sum_{e,f} \Phi_{0je} g^{ef} T_f \\ &= \sum_{e,f} g_{je} G^{ef} T_f \\ &= T_j\end{aligned}$$

□

We can extend the definition of the quantum cup product $\mathbb{Q}[[y]]$-linearly to the $\mathbb{Q}[[y]]$-module

$$A^*(X) \otimes_{\mathbb{Z}} \mathbb{Q}[[y]]$$

making it a $\mathbb{Q}[[y]]$-algebra. The most important property however is associativity, which we now prove.

6.2.1 Associativity of the Quantum Product

First, we need a lemma called the splitting lemma.

Recall from Lemma 3.27 we have the isomorphism

$$\overline{M}_{0,A\cup\{x\}}(\mathbb{P}^r, d_A) \times_{\mathbb{P}^r} \overline{M}_{0,B\cup\{x\}}(\mathbb{P}^r, d_B) \xrightarrow{\sim} D(A, B; d_A, d_B)$$

In fact this result as presented in [6] is is more generally true on $\overline{M}_{0,n}(X, \beta)$, where the assumptions on X and β are as we declared in the previous chapter. More precisely, this result is the following: Suppose $A \cup B = [n]$ is a partition, then there is an associated boundary divisor $D(A, B; \beta_1, \beta_2)$ in $\overline{M}_{0,n}(X, \beta)$, where $\beta_1 + \beta_2 = \beta$ and β_1 and β_2 are effective. Then we have an isomorphism

$$\overline{M}_{0,A\cup\{\bullet\}}(X, \beta_1) \times_X \overline{M}_{0,B\cup\{\bullet\}}(X, \beta_2) \xrightarrow{\sim} D((A, B; \beta_1, \beta_2))$$

We will not substantiate the details of these claims. They can be found in [6].

Lemma 6.7 (Splitting lemma) *Let ι denote the natural inclusion of $D(A, B; \beta_1, \beta_2)$ in the cartesian product $\overline{M}_{0,A\cup\{\bullet\}}(X, \beta_1) \times \overline{M}_{0,B\cup\{\bullet\}}(X, \beta_2)$, and let α be the embedding of $D(A, B; \beta_1, \beta_2)$ as a divisor in $\overline{M}_{0,n}(X, \beta)$, with $\beta = \beta_1 + \beta_2$. Then for any classes $\gamma_1, \ldots, \gamma_n \in A^*(X)$, we have*

$$\iota_* \circ \alpha^*(\mathrm{ev}_1^*(\gamma_1) \cup \cdots \cup \mathrm{ev}_n^*(\gamma_n)) =$$

$$\sum_{e,f} g^{ef} \left(\prod_{a\in A} \mathrm{ev}_a^*(\gamma_a) \cdot \mathrm{ev}_\bullet^*(T_e) \right) \times \left(\prod_{b\in B} \mathrm{ev}_b^*(\gamma_b) \cdot \mathrm{ev}_\bullet^*(T_f) \right)$$

Proof For notational simplicity, let

$$M = \overline{M}_{0,n}(X, \beta), \quad M_1 = \overline{M}_{0,A\cup\{\bullet\}}(X, \beta_1), \quad M_2 = \overline{M}_{0,B\cup\{\bullet\}}(X, \beta_2)$$

and

$$D = D(A, B; \beta_1, \beta_2)$$

Thus we have an isomorphism $D \cong M_1 \times_X M_2$. We have a commutative diagram

$$\begin{array}{ccccc} M & \xleftarrow[\alpha]{} & D & \xrightarrow{\iota} & M_1 \times M_2 \\ \Big\downarrow{\scriptstyle \underline{\mathrm{ev}}} & & \Big\downarrow{\scriptstyle \theta} & & \Big\downarrow{\scriptstyle \mathrm{ev}'} \\ X^n & \xleftarrow[p]{} & X^{n+1} & \xrightarrow{\delta} & X^{n+2} \end{array}$$

where $\underline{\mathrm{ev}}$ is the product of evaluation maps as before, θ is the product of the evaluation maps together with evaluation on the intersection point, ev' is the product of $\underline{\mathrm{ev}}$ and evaluation on the intersection point, separately on the A-twig and B-twig, δ is the diagonal embedding that repeats the last factor, and p is the projection that forgets the last factor. Then, utilizing the commutativity of the diagram, and pushing forward and pulling pack of cycle classes, we have

$$\begin{aligned} \iota_* \circ \alpha^*(\mathrm{ev}_1^*(\gamma_1) \cup \cdots \mathrm{ev}_n^*(\gamma_n)) &= \iota_* \alpha^* \circ \underline{\mathrm{ev}}^*(\gamma_1 \times \cdots \times \gamma_n) \\ &= \iota_* \theta^* \circ p^*(\gamma_1 \times \cdots \times \gamma_n) \\ &= \iota_* \circ \theta^*(\gamma_1 \times \cdots \times \gamma_n \times [X]) \\ &= \mathrm{ev}'^* \circ \delta_*(\gamma_1 \times \cdots \times \gamma_n \times [X]) \\ &= \mathrm{ev}'^*(\gamma_1 \times \cdots \gamma_n \times [\Delta]) \\ &= \sum_{e,f} g^{ef}\, \mathrm{ev}'^*(\gamma_1 \times \cdots \times \gamma_n \times T_e \times T_f) \end{aligned}$$

$$= \sum_{e,f} g^{ef} \left(\prod_{a \in A} \mathrm{ev}_a^*(\gamma_a) \cdot \mathrm{ev}_\bullet^*(T_e) \right) \times \left(\prod_{b \in B} \mathrm{ev}_b^*(\gamma_b) \cdot \mathrm{ev}_\bullet^*(T_f) \right)$$

Where Δ is the diagonal in $X \times X$, and using the definition of (g^{ij}), the class of the diagonal in $A^*(X \times X) = A^*(X) \otimes A^*(X)$ is

$$[\Delta] = \sum_{e,f} g^{ef} T_e \otimes T_f$$

□

Corollary 6.8 *Fix distinct integers $q, r, s, t \in [n]$, then*

$$\begin{aligned} G(q, r, \mid s, t) &:= \sum \int_D \mathrm{ev}_1^*(\gamma_1) \cup \cdots \cup \mathrm{ev}_n^*(\gamma_n) \\ &= \sum g^{ef} I_{\beta_1} \left(\prod_{a \in A} \gamma_a \cdot T_e \right) \cdot I_{\beta_2} \left(\prod_{b \in B} \gamma_b \cdot T_f \right) \end{aligned}$$

where the sum is over all partitions $A \cup B = [n]$ *such that* $q, r \in A$ *and* $s, t \in B$, $\beta_1 + \beta_2 = \beta$, *and* $0 \leq e, f \leq m$

Theorem 6.9 (Associativity) *The quantum product makes* $A^*(X) \otimes_{\mathbb{Z}} \mathbb{Q}[[y]]$ *into a commutative, associative* $\mathbb{Q}[[y]]$*-algebra with unit* T_0.

Proof We first write down what associativity means:

$$(T_i * T_j) * T_k = (\sum_{e,f} \Phi_{ije} g^{ef} T_f) * T_k = \sum_{e,f} \sum_{c,d} \Phi_{ije} g^{ef} \Phi_{fkc} g^c d T_d \tag{6.2}$$

$$T_i * (T_j * T_k) = (\sum_{e,f} \Phi_{jke} g^{ef} T_i) * T_f = \sum_{e,f} \sum_{c,d} \Phi_{jke} g^{ef} \Phi_{ifc} g^c d T_d \tag{6.3}$$

Equating the two equation, and using that the matrix (g^{cd}), transitivity is equivalent to

$$\sum_{e,f} \Phi_{ije} g^{ef} \Phi_{fkl} = \sum_{e,f} \Phi_{jke} g^{ef} \Phi_{ifl} \tag{6.4}$$

for all l. These differential equations are called the *Witten-Dijkgraaf-Verlinde-Verlinde (WDVV) equations*.

Now by Lemma 6.3, Φ_{ije} is the generating function for the invariants $I(\boldsymbol{h}^\nu \cdot T_i \cdot T_j \cdot T_e)$. Therefore by 6.1, we can re-write the left hand side, a product of two generating functions, as

$$\sum_{e,f} \sum_{n_A + n_B = n} \frac{n! \boldsymbol{y}^{\boldsymbol{n}}}{n_A! n_B!} g^{ef} I(\boldsymbol{h}^{\boldsymbol{n}_A} \cdot T_i \cdot T_j \cdot T_e) I(\boldsymbol{h}^{\boldsymbol{n}_B} \cdot T_i \cdot T_j \cdot T_e)$$

thus the WDVV equations become

$$\sum_{e,f} \sum_{n_A + n_B = n} \frac{n! \boldsymbol{y}^{\boldsymbol{n}}}{n_A! n_B!} g^{ef} I(\boldsymbol{h}^{\boldsymbol{n}_A} \cdot T_i \cdot T_j \cdot T_e) I(\boldsymbol{h}^{\boldsymbol{n}_B} \cdot T_i \cdot T_j \cdot T_e)$$

$$= \sum_{e,f} \sum_{n_A + n_B = n} \frac{n! \boldsymbol{y}^{\boldsymbol{n}}}{n_A! n_B!} g^{ef} I(\boldsymbol{h}^{\boldsymbol{n}_B} \cdot T_j \cdot T_k \cdot T_e) I(\boldsymbol{h}^{\boldsymbol{n}_B} \cdot T_f \cdot T_i \cdot T_l) \tag{6.5}$$

We claim that this equality is the direct result of the fundamental relation 3.5. By the fundamental relation, we have the equality

$$D(ij \mid kl) := \sum_{\substack{A \cup B = S \\ i,j \in A \\ k,l \in B \\ d_A + d_B = d}} D(A, B; d_A, d_B) \equiv \sum_{\substack{A \cup B = S \\ i,k \in A \\ j,l \in B \\ d_A + d_B = d}} D(A, B; d_A, d_B) =: D(jk \mid il)$$

Consider the classes T_i, T_j, T_k, T_l. We will take their pullbacks along the evaluations maps $\mathrm{ev}_i, \mathrm{ev}_j, \mathrm{ev}_k, \mathrm{ev}_l$. In addition, we will take the $(n-4)$-fold pullback

$$\underline{\mathrm{ev}}^{*(n-4)}(\underline{\gamma})$$

which is the $(n-4)$-fold cup product of $\mathrm{ev}^*(\gamma)$, where

$$\gamma = \sum y_i T_i$$

We will integrate these classes over the equivalent boundary divisors to get the equality

$$\int_{D(ij|kl)} \underline{\mathrm{ev}}^{*(n-4)}(\underline{\gamma}) \cup \mathrm{ev}_i^*(T_i) \cup \mathrm{ev}_i^*(T_j) \cup \mathrm{ev}_i^*(T_k) \cup \mathrm{ev}_i^*(T_l)$$

$$= \int_{D(jk|il)} \underline{\mathrm{ev}}^{*(n-4)}(\underline{\gamma}) \cup \mathrm{ev}_i^*(T_i) \cup \mathrm{ev}_i^*(T_j) \cup \mathrm{ev}_i^*(T_k) \cup \mathrm{ev}_i^*(T_l)$$

Let us expand the left-hand side.

Using Corollary 6.8, using the case $\gamma_i = \gamma$ for $0 \leq i \leq n-4$, $\gamma_{n-3} = T_i, \gamma_{n-2} = T_j, \gamma_{n-1} = T_k, \gamma_n = T_l$; and $q = n-3, r = n-2, s = n-1, t = n$, the left-hand side becomes

$$\sum_{e,f} \sum_{n_A+n_B=n} \binom{n-4}{n_A-2} \boldsymbol{y}^{\boldsymbol{n}} g^{ef} I_{\beta_1} \left(\prod_{a \in A} T_a \cdot T_1 \cdot T_k \cdot T_e \right) \cdot I_{\beta_2} \left(\prod_{b \in B} T_b \cdot T_k \cdot T_l \cdot T_f \right)$$

$$= \sum_{e,f} \sum_{n_A+n_B=n} \frac{\boldsymbol{n}!\boldsymbol{y}^{\boldsymbol{n}}}{n_A! n_B!} g^{ef} I_{\beta_1} \left(\prod_{a \in A} T_a \cdot T_1 \cdot T_k \cdot T_e \right) \cdot I_{\beta_2} \left(\prod_{b \in B} T_b \cdot T_k \cdot T_l \cdot T_f \right)$$

where the sum is over all $\beta_1 + \beta_2 = \beta$, and $n_1 + n_2 = n$. □

6.3 Proof of Kontsevich's Formula via Quantum Cohomology

For this section we fix $X = \mathbb{P}^m$ for simplicity, although much of the content can be done with general homogeneous X, with slight modifications. We also fix the basis T_0 as the fundamental class, T_1 as the point class, T_2 the class of a line, etc. This means in particular that the numbers g_{ij} is such that

$$g_{ef} = \begin{cases} 1 & e+f=m \\ 0 & \text{otherwise} \end{cases}$$

therefore the quantum cup product is simpler to describe:

$$T_i * T_j = \sum_{e+f=m} \Phi_{ije} T_f$$

We will now break the Gromov-Witten potential into the sum of two generating functions:

$$\Phi(\boldsymbol{y}) = \Phi^{\text{classical}}(\boldsymbol{y}) + \Gamma(\boldsymbol{y})$$

where the classical part are the terms for $\beta = 0$:

$$\Phi^{\text{classical}}(y) = \sum_{n_0+\cdots+n_m\geq 3} \frac{y_0^{n_0}}{n_0!}\cdots\frac{y_m^{n_m}}{n_m!} I_0(T_0^{n_0}\cdots T_m^{n_m})$$
$$= \sum_{n_0+\cdots+n_m\geq 3} \frac{y_0^{n_0}}{n_0!}\cdots\frac{y_m^{n_m}}{n_m!} \int_X (T_0^{n_0}\cup\cdots\cup T_m^{n_m})$$
$$= \sum_{i,k,j} \frac{y_i y_j y_k}{3!} I_0(T_i\cdot T_j\cdot T_k)$$

Then clearly

$$\Phi_{ijk}^{\text{classical}} = I_0(T_i\cdot T_j\cdot T_k)$$

Decomposing the quantum cup product Now we can decompose the quantum cup product into the classical part and the quantum part:

$$T_i * T_j = \sum_{e+f=m} \Phi_{ije} T_f$$
$$= \sum_{e+f=m} (I_0(T_i\cdot T_j\cdot T_e) + \Gamma_{ije}) T_f$$
$$= (T_i \cup T_j) + \sum_{e+f} \Gamma_{ije} T_f$$

Now we will fix our attention to the case of the complex projective plane. So fix $m = 2$. Then the basis elements for $A^*(\mathbb{P}^2)$ are $T_0 = 1$ (i.e. the fundamental class), T_1 is the class of a line, and T_2 is the class of a point. Using the decomposition above, we can write down explicitly the multiplication for the quantum cup product as

$$T_i * T_j = \Phi_{ij0}T_2 + \Phi_{ij1}T_1 + \Phi_{ij2}T_0$$

By the associative property the following two equations must be equal:

$$(T_1 * T_1) * T_2 = (\Gamma_{221}T_1 + \Gamma_{222}T_0) + \Gamma_{111}(\Gamma_{121}T_1 + \Gamma_{122}T_0) + \Gamma_{112}T_2$$

$$T_1 * (T_1 * T_2) = \Gamma_{121}(T_2 + \Gamma_{111}T_1 + \Gamma_{112}T_0) + \Gamma_{122}T_1$$

In particular the equality of the coefficients of T_0 gives

$$\Gamma_{222} = \Gamma_{122}^2 - \Gamma_{111}\Gamma_{122} \tag{6.6}$$

Now we consider the partial derivatives of Γ:

$$\Gamma_{ijk}(y) = \sum_{\nu} \frac{\boldsymbol{y}^{\nu}}{\boldsymbol{\nu}!} I_+(\boldsymbol{h}^{\nu}\cdot T_i\cdot T_j\cdot T_k)$$

We consider in particular the case $y_0 = y_1 = 0$. Then

$$\Gamma_{ijk}(y) = \sum \frac{y_2^n}{n!} I_+((T_2)^n \cdot T_i \cdot T_i \cdot T_k)$$

which is generating function for the numbers $I_+((T_2)^n \cdot T_i \cdot T_i \cdot T_k)$. Thus using the product rule 6.1, we can rewrite 6.6 as

$$I_+((T_2)^n \cdot T_2 \cdot T_2 \cdot T_2) = \sum_{n_A + n_A = n} \frac{n!}{n_A! n_B!} I_+((T_2)^{n_A} \cdot T_1 \cdot T_1 \cdot T_2) I_+((T_2)^{n_B} \cdot T_1 \cdot T_1 \cdot T_2)$$

$$- \sum_{n_A + n_A = n} \frac{n!}{n_A! n_B!} I_+((T_2)^{n_A} \cdot T_1 \cdot T_1 \cdot T_1) I_+((T_2)^{n_B} \cdot T_1 \cdot T_2 \cdot T_2)$$

Now let us examine the (positive)collected Gromov-Witten invariants $I_+((T_2)^n \cdot T_i \cdot T_j \cdot T_k)$. Recall the definition

$$I_+((T_2)^n \cdot T_i \cdot T_j \cdot T_k) = \sum_{d=1}^{\infty} I_d((T_2)^n \cdot T_i \cdot T_j \cdot T_k)$$

However, only compatible d and n give contribution. Notice these Gromov-Witten invariants all have $n+3$ marks, so we are working in $\overline{M}_{0,n+3}(\mathbb{P}^2, d)$. Recall from Proposition 3.29, the dimension of this space is $3d + 2 + n$. We also need to check the sum of the codimensions of the classes, which is $2n + i + j + k$, since T_2 is the point class. Now only an n such that these two numbers are equal can there be a non-zero contribution:

$$n = 3d + 2 - i - j - k$$

Before be go any further, a quick lemma on the Gromov-Witten invariant on $\mathbb{P}^r$:

Lemma 6.10 *In $\overline{M}(\mathbb{P}^r, d)$, suppose $d > 0$ and that one of the input classes h in the Gromov-Witten invariant*

$$I_d(\gamma_1 \cdots \gamma_n \cdot h)$$

is the hyperplane class, then

$$I_d(\gamma_1 \cdots \gamma_n \cdot h) = d \cdot I_d(\gamma_1 \cdots \gamma_n)$$

Proof We denote

$$\hat{\text{ev}}_i \colon \overline{M}_{0,n+1}(\mathbb{P}^r, d) \to \mathbb{P}^r$$

the evaluation map. The hat is placed here to differentiate it from the evaluation map

$$\text{ev}_i \colon \overline{M}_{0,n}(\mathbb{P}^r, d) \to \mathbb{P}^r$$

and to we have the commutative diagram

$$\begin{array}{ccc} \overline{M}_{0,n+1}(\mathbb{P}^r, d) & \xrightarrow{\hat{\text{ev}}_i} & \mathbb{P}^r \\ \downarrow \varepsilon & \nearrow_{\text{ev}_i} & \\ \overline{M}_{0,n}(\mathbb{P}^r, d) & & \end{array}$$

where ε is the forgetful map.

Consider the class $\hat{\text{ev}}_{n+1}^*$ in $\overline{M}_{0,n+1}(\mathbb{P}^r, d)$. It is the class of $\hat{\text{ev}}_{n+1}^{-1}(H)$ for some hyperplane in $\mathbb{P}^r$. By definition of the inverse, it is also the locus of maps whose $n+1$-th marked point p_{n+1} is mapped to H. Now the forgetful map ε restricted to $\hat{\text{ev}}_{n+1}^{-1}(H)$:

$$\varepsilon|_{\hat{\text{ev}}_{n+1}^{-1}(H)} \colon \hat{\text{ev}}_{n+1}^{-1}(H) \to \overline{M}_{0,n}(\mathbb{P}^r, d)$$

is generically of degree d.

Now we have

$$\begin{aligned} I_d(\gamma_1 \cdots \gamma_n \cdot h) &= \int_{\overline{M}_{0,n+1}(\mathbb{P}^r,d)} \hat{\text{ev}}_1{}^*(\gamma_1) \cup \cdots \cup \hat{\text{ev}}_n{}^*(\gamma_n) \cup \hat{\text{ev}_{n+1}}{}^*(h) \\ &= \int_{\overline{M}_{0,n+1}(\mathbb{P}^r,d)} \underline{\hat{\text{ev}}}^*(\underline{\gamma}) \cup \hat{\text{ev}}_{n+1}^*(h) \\ &= \int_{[\hat{\text{ev}}_{n+1}^*{}^{-1}(H)]} \underline{\hat{\text{ev}}}^*(\underline{\gamma}) \\ &= \int_{\varepsilon_*[\hat{\text{ev}}_{n+1}^{-1}(H)]} \underline{\hat{\text{ev}}}^*(\underline{\gamma}) \\ &= \int_{d[\overline{M}_{0,n}(\mathbb{P}^r,d)]} \underline{\hat{\text{ev}}}^*(\underline{\gamma}) \\ &= d \cdot I_d(\gamma_1 \cdots \gamma_n) \end{aligned}$$

□

Now back to where we were before. We re-write the relation

$$I_+((T_2)^n \cdot T_2 \cdot T_2 \cdot T_2) = \sum_{n_A+n_A=n} \frac{n!}{n_A! n_B!} I_+((T_2)^{n_A} \cdot T_1 \cdot T_1 \cdot T_2) I_+((T_2)^{n_B} \cdot T_1 \cdot T_1 \cdot T_2)$$

$$- \sum_{n_A+n_A=n} \frac{n!}{n_A! n_B!} I_+((T_2)^{n_A} \cdot T_1 \cdot T_1 \cdot T_1) I_+((T_2)^{n_B} \cdot T_1 \cdot T_2 \cdot T_2) \tag{6.7}$$

with only the contributing n, i.e. ones that satisfy

$$n = 3d + 2 - i - j - k$$

We will also use Lemma 6.10 to take out the T_1 factors in the Gromov-Witten invariants to become d's. Then recall from Proposition 5.3 that

$$N_d = I_d((T_2)^{3d-1})$$

Then we have the following five substitutions for each Gromov-Witten invariant:

For the first term $I_+((T_2)^n \cdot T_2 \cdot T_2 \cdot T_2)$, we have $n = 3d + 2 - 2 - 2 - 2 = 3d - 4$, so

$$\begin{aligned} I_+((T_2)^n \cdot T_2 \cdot T_2 \cdot T_2) &= I_d((T_2)^{3d-4} \cdot T_2 \cdot T_2 \cdot T_2) \\ &= I_d((T_2)^{3d-1}) \\ &= N_d \end{aligned}$$

For the term $I_+((T_2)^{n_A} \cdot T_1 \cdot T_1 \cdot T_2)$, we have $n_A = 3d_A + 2 - 1 - 1 - 2 = 3d_A - 2$, so

$$\begin{aligned} I_+((T_2)^{n_A} \cdot T_1 \cdot T_1 \cdot T_2) &= I_{d_A}((T_2)^{3d_A-2} \cdot T_1 \cdot T_1 \cdot T_2) \\ &= d_A^2 I_{d_A}((T_2)^{3d_A-1}) \\ &= d_A^2 N_{d_A} \end{aligned}$$

For the term $I_+((T_2)^{n_B} \cdot T_1 \cdot T_1 \cdot T_2)$, we have $n_B = 3d_B + 2 - 1 - 1 - 2 = 3d_B - 2$, so

$$\begin{aligned} I_+((T_2)^{n_B} \cdot T_1 \cdot T_1 \cdot T_2) &= I_{d_B}((T_2)^{3d_B-2} \cdot T_1 \cdot T_1 \cdot T_2) \\ &= d_B^2 I_{d_A}((T_2)^{3d_B-1}) \\ &= d_B^2 N_{d_B} \end{aligned}$$

For the term $I_+((T_2)^{n_A} \cdot T_1 \cdot T_1 \cdot T_1)$, we have $n_A = 3d_A + 2 - 1 - 1 - 1 = 3d_A - 1$, so

$$\begin{aligned} I_+((T_2)^{n_A} \cdot T_1 \cdot T_1 \cdot T_1) &= I_{d_A}((T_2)^{3d_A-1} \cdot T_1 \cdot T_1 \cdot T_1) \\ &= d_A^3 I_{d_A}((T_2)^{3d_A-1}) \\ &= d_A^3 N_{d_A} \end{aligned}$$

For the term $I_+((T_2)^{n_B} \cdot T_1 \cdot T_2 \cdot T_2)$, we have $n_B = 3d_B + 2 - 1 - 2 - 2 = 3d_B - 3$, so

$$\begin{aligned} I_+((T_2)^{n_B} \cdot T_1 \cdot T_2 \cdot T_2) &= I_{d_B}((T_2)^{3d_B-3} \cdot T_1 \cdot T_2 \cdot T_2) \\ &= d_B I_{d_A}((T_2)^{3d_B-1}) \\ &= d_B N_{d_B} \end{aligned}$$

And therefore, we can finally re-write 6.7 as

$$
\begin{aligned}
N_d &= \sum_{d_A+d_B=d} \frac{(3d-4)!}{(3d_A-2)!(3d_B-2)!} d_A^2 N_{d_A} d_B^2 N_{d_B} - \sum_{d_A+d_B=d} \frac{(3d-4)!}{(3d_A-1)!(3d_B-3)!} d_A^3 N_{d_A} d_B N_{d_B} \\
&= \sum_{d_A+d_B=d} \binom{3d-4}{3d_A-2} d_A^2 N_{d_A} d_B^2 N_{d_B} - \sum_{d_A+d_B=d} \binom{3d-4}{3d_A-1} d_A^3 N_{d_A} d_B N_{d_B} \\
&= \sum_{d_A+d_B=d} N_{d_A} N_{d_B} d_A^2 d_B \left(d_B \binom{3d-4}{3d_A-1} - d_A \binom{3d-4}{3d_A-1} \right)
\end{aligned}
$$

which is Kontsevich's formula.

Appendix A

Intersection Theory

A.1 Rational Equivalence

Definition A.1 (Hartshorne pg.16) Let Y be a variety. We denote by $\mathcal{O}(Y)$ the ring of all regular functions on Y. If P is a point of Y, we define the *local ring of P on Y*, $\mathcal{O}_{P,Y}$ (or simply $\mathcal{O}_P$) to be the ring of germs of regular functions on Y, near P. In other words, an element of $\mathcal{O}_P$ is a pair $< U, f >$ where U is an open subset of Y containing P, and f is a regular function on U, and where we identify two such pairs (U, f) and (V, f) if $f = g$ on $U \cap V$.

It is important to know indeed $\mathcal{O}_P$ is a local ring, i.e. that it has a unique maximal ideal. Its maximal ideal $\mathfrak{m}$ is the set of germs/equivalence classes $[(U, g)]$ such that g vanishes at P. Indeed the complement of $\mathfrak{m}$ is the set of all units: if $f(P) \neq 0$, then $1/f$ is a regular function in some neighborhood of P.

Another fact about the maximal ideal is that the residue field $\mathcal{O}_P/k$ is isomorphic to the ground field k. This can be seen from the following short exact sequence:

$$0 \to \mathfrak{m} \hookrightarrow \mathcal{O}_P \xrightarrow{f(p)} k \to 0$$

Definition A.2 (Hartshorne pg.15) If Y is a variety, we define the *function field* $R(Y)$ of Y as follows: an element of $R(Y)$ is an equivalence class of of pairs (U, f) where U is a nonempty open subset of Y, f is a regular function on U, and where we identify two pairs (U, f) and (V, g) if $f = g$ on $U \cap V$. The elements of $R(Y)$ are called *rational functions* on Y.

Note that $R(Y)$ is indeed a field.

Proposition A.3 (The Local Ring of a Subvariety. Hartshorne Excercise 3.13) *Let $Y \subset X$ be a subvariety. Let $\mathcal{O}_{Y,X}$ be the set of equivalence classes $[(U, f)]$ where*

$U \subset X$ is open, $U \cap Y \neq \emptyset$, and f is a regular function on U. The equivalence relation is the following: $(U, f) \sim (V, g)$ is $f = g$ on $U \cap V$. Then $\mathcal{O}_{Y,X}$ is a local ring, with residue field $R(Y)$ and dimension $= \dim X - \dim Y$. It is called the local ring of Y on X.

Proof

Note that this is just a generalization of the local ring at a point P: If $Y = P$ is a point we just get $\mathcal{O}_P$. Also, if $Y = X$ we get $R(X)$. Note also that if Y is not a point, then $R(Y)$ is not algebraically closed, thus in this way we get residue fields which are not algebraically closed.

Consider a variety X, and a subvariety V of codimension 1. Then the local ring $A = \mathcal{O}_{V,X}$ is a one-dimensional local domain. Let $r \in R(X)^*$ (i.e. $R(X) \setminus 0$, the multiplicative group), we will define the *order of vanishing of r along V* as the following:

$$\mathrm{ord}_V(r) := l_A \left(A \big/ (r) \right)$$

where l_A denotes the length of the A-module in parentheses.

Recall what the length of a module is:

Definition A.4 (Length of a module) Let M be a module over a ring R. Given a chain of submodules of the following form

$$N_0 \subsetneq N_1 \subsetneq \cdots \subsetneq N_n$$

we say that the chain has length n. The *length of the module M* is defined to be the supremum of the length of all of its chains.

A.1.1 Cycles and Rational Equivalence

Let X be an algebraic scheme. A k-cycle on X is a finite formal sum

$$\sum n_i [V_i]$$

where the V_i are k-dimensional subvarieties of X, and the n_i are integers.

We can put a group structure on the set of all k-cycles on X, this group is denoted $Z_k X$. The group operation is simply addition of finite linear combinations to get another finite linear combination. The additive identity is just 0 (times any subvariety?), and the inverses are the obvious ones. This group can also be described as the free abelian group on the k-dimensional subvarieties of X. To each subvariety V we can consider itself as an element of the group (just trivially with 1 as the coefficient), we will denote $[V]$ when considered as an element of the group.

For any $(k+1)$-dimensional subvariety W of X, and any $r \in R(W)^*$, define a k-cycle $[\operatorname{div}(r)]$ on X by

$$[\operatorname{div}(r)] := \sum \operatorname{ord}_V(r)[V]$$

the sum is taken over all codimension one subvarieties V of W.

A k-cycle $\alpha \in Z_k X$ is *rationally equivalent to zero*, written $\alpha \sim 0$, if there are a finite number of $(k+1)$-dimensional subvarieties W_i of X, and $r_i \in R(W_i)^*$, such that

$$\alpha = \sum [\operatorname{div}(r_i)]$$

The set of cycles rationally equivalent to zero form a subgroup $\operatorname{Rat}_k X$ of $Z_k X$. This can be readily observed from the fact that

$$[\operatorname{div}(r^{-1})] = -[\operatorname{div}(r)]$$

thus the inversion is closed in the subgroup.

The group of *k-cycles modulo rational equivalence*, or the *Chow group of k-dimensional cycles* is the quotient

$$A_k X = {Z_k X} / {\operatorname{Rat}_k X}$$

A.2 Pushforward of Cycles

First, we define the notion of a proper morphism of schemes.

Definition A.5 Let $f : X \to Y$ be a morphism of schemes. Write

$$\Delta : X \to X \times_Y X$$

for the diagonal morphism, i.e. the natural morphism in the category of schemes

$$\Delta : X \xrightarrow{(Id,Id)} X \times X$$

We say that the morphism f is *separated* if $\Delta(X)$ is a closed subscheme of $X \times_Y X$, i.e. the diagonal map is a closed immersion.

We say that a scheme X is separated if the unique morphism

$$X \to \operatorname{Spec}(\mathbb{Z})$$

is separated.

This is an analogue of Hausdorffness. There is an equivalent definition of a topological space being Hausdorff if the diagonal $\Delta = \{(x,x) : x \in X\}$ is a closed subset of the product topological space $X \times X$.

Definition A.6 Let $f : X \to Y$ be a morphism of schemes. We say that f is a *finite* morphism if Y has an open cover by affine shcemes $V_i = \operatorname{Spec} B_i$ such that for each i,

$$f^{-1}(V_i) = U_i$$

is an open affine subscheme $\operatorname{Spec} A_i$ (viewed as an open embedding), and the restriction of f to U_i, which induces a ring homomorphism

$$B_i \to A_i$$

makes A_i a finitely generated module over B_i

We say that f is a *finite type* morphism if

$$f^{-1}(V_i) = U_i$$

has a finite covering by affine open subschemes $U_{ij} = \operatorname{Spec} B_{ij}$ with B_{ij} being an A_i-algebra of finite type (i.e. finitely generated as an A_i algebra).

Definition A.7 Let $f : X \to Y$ be a morphism of schemes. We say that f is *universally closed* if for every scheme Z with a morphism $Z \to Y$, the projection from the fiber product

$$X \times_Y Z \to Z$$

is a closed map of the underlying topological spaces.

And finally,

Definition A.8 Let $f : X \to Y$ be a morphism of schemes. We say that f is a *proper* morphism if it is separated, of finite type, and universally closed.

We need the following fact which we will not prove right now:

Proposition A.9 *Let $f : X \to Y$ be a proper morphism. Then for any subvariety V of X, the image $W = f(V)$ is a closed subvariety of Y.*

In such a case (proper morphism), we get an imbedding of fields

$$R(W) \hookrightarrow R(V)$$

To justify this, first consider a dominant morphism of varieties $h : A \to B$, i.e. a morphism in which the image is dense (this is trivially true in our case from V to $W = f(V)$). Suppose $\varphi \in R(B)$ is a rational function on B, thus by definition it is an equivalence class $[(U, g \in \mathcal{O}(U))]$, under the familiar equivalence relation. Pick a representative (U, g) for φ. Since $f(A)$ is dense, $f^{-1}(U)$ is non-empty. Hence $[(f^{-1}(U), f \circ g)]$ is a rational function on X. We see that equivalent functions pullback to equivalent function. In this way we obtain an embedding $R(B) \hookrightarrow R(A)$.

Returning to the case of Proposition A.9, it is a fact from [3] that $R(V)/R(W)$ is a finite field extension if W has the same dimension as V. Thus we can set

$$\deg(V/W) := \begin{cases} [R(V) : R(W)] & \text{if } \dim(W) = \dim(V) \\ 0 & \text{if } \dim(W) < \dim(V) \end{cases}$$

uppose V is k-dimensional in X, we can also define a k-cycle in Y to be

$$f_*[V] = \deg(V/W)[W]$$

We can extend linearly to define a ring homomorphism, called the *push-forward*

$$f_* : Z_k X \to Z_k Y$$

Theorem A.10 *If $f : X \to Y$ is a proper morphism, and α is a k-cycle on X which is rationally equivalent to zero, then $f_*\alpha$ is rationally equivalent to zero on Y.*

A.3 Flat Pullback of Cycles

Definition A.11 A morphism of schemes $f : X \to Y$ is *flat* if the induced morphism on stalks at every $P \in X$:

$$f_P^\# : \mathcal{O}_{Y,f(P)} \to \mathcal{O}_{X,P}$$

is a flat morphism of rings, i.e. this morphism makes $\mathcal{O}_{X,P}$ a flat $\mathcal{O}_{Y,f(P)}$-module. (Recall that a morphism of schemes has an underlying morphism of sheaves on Y

$$f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X$$

which induces a local ring homomorphism of stalks

$$f_P^\# : \mathcal{O}_{Y,f(P)} \to \mathcal{O}_{X,P}$$

in which the locality is required as part of the definition of the morphism of a locally ringed space.)

There is a geometrical interpretation for flatness. Roughly, it is a "smoothly varying family of fibers.

Definition A.12 A morphism of schemes $f : X \to Y$ has relative dimension n if for all subvarieties V of Y, and all irreducible components V' of $f^{-1}(V)$, $\dim V' = \dim V + n$.

Fact: If f is flat, Y is irreducible, and X has pure dimension equal to $\dim Y + n$, then f has relative dimension n, and all base extensions $X \times_Y Y' \to Y'$ have relative dimesion n.

We assume every flat morphism to have a relative dimension n for some integer n.

For a flat morphism $f : X \to Y$, and any subvariety V of Y, set

$$f^*[V] = [f^{-1}(V)]$$

where $f^{-1}(V)$ is the inverse image scheme with scheme structure given by fiber products, which is a subscheme of X of pure dimension $\dim(V) + n$ (from flatness), and $[f^{-1}(V)]$ is this subscheme's cycle (of schemes). We can extend by linearity to *flat pullback homomorphisms* (of rings)

$$f^* : Z_k Y \to Z_{k+n} X$$

Lemma A.13 *If $f : X \to Y$ is flat, then for any subscheme Z of Y,*

$$f^*[Z] = [f^{-1}(Z)]$$

Theorem A.14 *Let $f\colon X \to Y$ be a flat morphism of relative dimension n, and α a k-cycle on Y which is rationally equivalent to zero. Then $f^*\alpha$ is rationally equivalent to zero in $Z_{k+n}X$. Therefore there are induced homomorphisms, called the* flat pull-backs

$$f^*\colon A_k Y \to A_{k+n} X$$

so that A_* becomes a contravariant functor for flat morphisms.

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