Topics in Symplectic Topology

ETH

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Contents

Chapter 1

Preliminaries

Textbooks: McDuff-Salomon: Introduction to Symplectic Topology, 3rd Edition; Polterovich: The Geometry of the Group of Hamiltonian Diffeomorphisms; Polterovich-Rosen: Function Theory on Symplectic Manifolds; Y.G. Oh: Symplectic Topology and Floer Homology.

Chapter 2

Motivation for the Study of Symplectic Structures

2.1 Motivation 1: Hamiltonian Dynamics/Classical Mechanics

In Newtonian mechanics, we consider $\mathbb{R}^n$ with coordinates $q = (q_1, \ldots, q_n)$. Consider a mechanical system of the following type: P is a particule of mass m and it moves under a field of potential forces

$$\vec{F}(q) = -\boldsymbol{\nabla} U$$

and

$$U : \mathbb{R}^n \to \mathbb{R}^n$$

Newton's Law asserts:

$$\vec{F} = m\vec{a}$$

thus

$$m\ddot{q}(t) = -\boldsymbol{\nabla} U(q(t))$$

which is a second order O.D.E.

We put $p(t) := m\vec{v}(t) = m\dot{q}(t)$. Consider $\mathbb{R}^{2n}$ with coordinates $(q, p) = (1_1, \ldots, q_n, p_1, \ldots, p_n)$. Then the above becomes

$$\dot{p}(t) = -\boldsymbol{\nabla} U(q(t))$$

and

$$\dot{q}(t) = \frac{1}{m} p(t)$$

which is a system of first order O.D.E.'s. Written in coordinates:

Now define a fuction called the *Hamiltonian Function*:

$$\begin{aligned} H &: \mathbb{R}^{2n} \to \mathbb{R} \\ (q, p) &\mapsto \frac{m|v|^2}{2} + U(q) = \frac{1}{2m}|p|^2 + U(q) \end{aligned}$$

The system of equations becomes:

$$
\begin{aligned}
\dot{p}_1(t) &= -\frac{\partial H}{\partial q_1}(q(t), p(t)) \\
&\vdots \\
\dot{p}_n(t) &= -\frac{\partial H}{\partial q_n}(q(t), p(t)) \\
\dot{q}_1(t) &= \frac{\partial H}{\partial p_1}(q(t), p(t)) \\
&\vdots \\
\dot{q}_n(t) &= \frac{\partial H}{\partial p_n}(q(t), p(t))
\end{aligned}
$$

equivalently

$$
\begin{aligned}
\dot{p}(t) &= -\frac{\partial H}{\partial q}(q(t), p(t)) \\
\dot{q}(t) &= \frac{\partial H}{\partial p}(q(t), p(t))
\end{aligned}
$$

this is an example of a Hamiltonian system.

Flows

Let $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$. Consider our system of equations with initial conditions $q(0) = x, p(0) = y$. The solution $(q(t), p(t))$ depends smoothly on t. Denote

$$\Phi_t(x, y) \coloneqq (q(t), p(t))$$

the unique solution at time t with given initial conditions. We get a map called the *flow*:

$$
\begin{aligned}
\Phi_t \colon \mathbb{R}^{2n} &\to \mathbb{R}^{2n} \\
(x, y) &\mapsto \Phi_t(x, y)
\end{aligned}
$$

with the following properties:

1. $\Phi_0 = \mathrm{Id}$
2. Φ_t is a diffeomorphism for all t.

Question: Id Φ_t special?

Theorem 2.1.1 (Louisville)**.** *For every t, $\Phi_t \coloneqq \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is a volume preserving diffeomerphism.*

Volume preserving means:

$$\det(D\Phi_t) \equiv 1$$

Symplectc Structure

Define the following differential 2-form on the manifold $\mathbb{R}^{2n}$

$$\omega_{\text{std}} \coloneqq \mathrm{d}p_1 \wedge \mathrm{d}q_1 + \cdots + \mathrm{d}p_n \wedge \mathrm{d}q_n$$

so it eats two vectors and spits out a number.

Theorem 2.1.2 (Darbaux). *Φ_t preserves ω_{std}. In other words:*

$$\Phi_t * \omega_{std} = \omega_{std}$$

i.e. for every z in$\mathbb{R}^{2n}$ and every $u, v \in T_z(\mathbb{R}^{2n}) \cong \mathbb{R}^{2n}$ we have that

$$\omega_{std}|_{\Phi_t(z)}(D\Phi_t(u), D\Phi_t(v)) = \omega_{std}|_z(u, v)$$

Consider

$$vol - form_{\mathbb{R}^{2n}} = \frac{\omega_{std} \wedge \cdots \wedge \omega_{std}}{n!}$$

Then Darboux implies Liousville.

2.2 Motivation 2: Complex Differential Geometry/Hermitian Geometry

Consider $\mathbb{C}^n$ with standard Hermitian structure i.e. inner product

$$h(u, v) := \sum_{k=1}^{n} u_k \cdot \bar{v}_k$$

Writing the real and imginary parts $u_k = u'_k + iu''_k$ and $v_k = v'_k + iv''_k$ then

$$h(u, v) = \sum_{k=1}^{n} (u'_k v'_k + u''_k v''_k) + i \sum_{k=1}^{n} (-u'_k v''_k + u''_k v'_k)$$

Put

$$g(u, v) := \operatorname{Re} h(u, v)$$

and

$$-\operatorname{Im} h(u, v)$$

If we identify $\mathbb{C}^n \cong \mathbb{R}^{2n}$ by the following

$$(x_1 + iy_1, \ldots, x_n + iy_n) \leftrightarrow (x_1, \ldots, x_n, y_1, \ldots, y_n)$$

then $g(u, v)$ is the scalar product (u, v).

$$\Omega(u, v) = \omega_{std}(u, v)$$
$$h = g - i\omega$$

where the g is the Riemannian geometry and ω is the symplectic geometry.

Are there any symplectic invariants of domains? Let us fix $U, U' \subset \mathbb{R}^{2n}$ open domains (connected subsets). Suppose the U and U' are diffeomorphic.

Question: Does there exist a volume preserving diffeomorphism $\phi \colon U \to U'$? i.e.

$$\phi^* \mu_{vol} = \mu_{vol}$$

where μ_{vol} is the volume form. The answer is a non-trivial theorem:

Theorem 2.2.1. *Such a ϕ exits if and only if*

$$\operatorname{vol}(U) = \operatorname{vol}(U')$$

This theorem follows from a theorem by Mozer in the 1960s.

Question: Does ther exist a symplectic dffeoemorphism $\phi\colon U \to U'$ i.e.

$$\phi^* \omega_{std} = \omega_{std}$$

The answer is in general not known. It is necessary that

$$\mathrm{vol}(U) = \mathrm{vol}(U')$$

but this is far from being a sufficient condition.

Question: (Extension to symplectic embeddings) Does there exist $\phi\colon U \hookrightarrow U'$ such that

- ϕ is volume preserving?
- ϕ is symplectic ?

Theorem 2.2.2 (Non-Squeezing Theorem, Gromov 1985)**.** *Let $B^{2n}(R) = \{x \in \mathbb{R}^{2n} \mid |x| \leq R\}$ be the closed ball. There exists a symplectic embedding*

$$\phi\colon B^{2n}(R) \hookrightarrow B^2(r) \times \mathbb{R}^{2n-2}$$

if and only if $R \leq r$.

Definition 2.2.3. The *Gromov radius* is

$$\zeta_{Gr}(U) = \sup_R \{\exists \phi\colon B^{2n}(R) \hookrightarrow U \text{ symplectic embedding}\}$$

Symplectic Packing

Let $U \subset \mathbb{R}^{2n}$ be a domain. Fix a natural number $N \geq 1$. Consider symplectic embeddings

$$\phi\colon B^{2n}(r_1) \sqcup \cdots \sqcup B^{2n}(r_N) \hookrightarrow U$$

such that

$$\phi^* \omega_{std} = \omega_{std}$$

When does such a thing exists?

A necessary condition is

$$\mathrm{vol}\, B^{2n}(r_1) + \cdots + \mathrm{vol}\, B^{2n}(r_N) \leq \mathrm{vol}(U)$$

$$\frac{\pi^n}{n!}(r_1^{2n} + \dots r_N^{2n}) \leq \mathrm{vol}(U)$$

Take $U = B^{2n}(R)$. Then the volume inequality becomes

$$r_1^{2n} + r_N^{2n} \leq R^{2n}$$

Theorem 2.2.4 (Gromov 1985)**.** *If there exists a symplectic embedding*

$$\phi\colon B^{2n}(r_1) \sqcup B^{2n}(r_2) \hookrightarrow B^{2n}(R)$$

then

$$r_1^2 + r_2^2 \leq R^2.$$

In dimension 4, more can be said:

Theorem 2.2.5 (Gromov again)**.** *If there exists a symplectic embedding*

$$\phi\colon \sqcup_{k=1}^5 B^4(r_k) \hookrightarrow B^4(R)$$

then for every $i \neq j$, $r_i^2 + r_j^2 \leq R^2$, and

$$r_1^2 + \cdots + r_5^2 \leq 2.$$

This last inequality is in fact sharp, proven by McDuff and Polterovich in 1993.

Symplectic packing by equal balls: Define

$$v_N(U) \coloneqq \sup_r \frac{N \cdot \mathrm{vol}(B^2(r))}{\mathrm{vol}(U)} \in (0,1]$$

where the supremum is taken over all posible r such that ther exits symplectic embedding

$$\phi \colon B^{2n}(r) \sqcup \cdots \sqcup B^{2n}(r) \text{ (N times)} \hookrightarrow U$$

Theorem 2.2.6. *Take $U = B^4(R)$.*

N	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	...
v_N	*1*	*1/2*	*3/4*	*1*	*4/5*	*24/25*	*63/64*	*288/289*	*1*	*1*	*1...*

Chapter 3

Symplectic Manifolds

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3.1 Definition and First Examples

Recall from last time on the manifold $\mathbb{R}^{2n}$ with cooridnates $(q,p) = (q_1, \dots q_n, p_1, \dots, p_n)$ we defined the symplectic structure, i.e. a differential 2-form to be

$$\omega_{\text{std}} = \mathrm{d}p_1 \wedge \mathrm{d}q_1 + \cdots + \mathrm{d}p_n \wedge \mathrm{d}q_n$$

Definition 3.1.1. Let M^{2n} be a $2n$-dimensional smooth manifold. A *symplectic structure (or form)* on M is a differential 2-form ω which is locally diffeomorphic to ω_{std}, i.e. for every $x \in M$ there exists a neighborhood U_x of x and an open subset $V \subset \mathbb{R}^{2n}$ (depending on x) and a diffeomorphism $\phi \colon U_x \to V$ with $\phi^* \omega_{\text{std}} = \omega$. We call the pair (M, ω) a *symplectic manifold.*

This is a type of geometruc structure that has no local invariant, so all symplectic manifolds look the same locally if they are of the same dimension. This is an instance of the “locally flat” property.

Notice that ω_{std} is closed, i.e.

$$\mathrm{d}\omega_{\text{std}} = 0$$

Therefore ω is closed as well.

Also, ω_{std} is a non-degenerate form, i.e. for every $0 \neq u \in \mathbb{R}^{2n}$ there exists $v \in \mathbb{R}^{2n}$ such that $\omega_{\text{std}}(u,v) \neq 0$. Equivalently, the map

$$\begin{aligned} \mathbb{R}^{2n} &\to \left(\mathbb{R}^{2n}\right)^* \\ v &\mapsto \omega_{\text{std}}(u,v) \end{aligned}$$

is an isomorphism. Or yet equivalently, the matrix representing ω_{std} is invertible. Therefore we conclude that ω is also non-degenerate. Thus for every $x \in M$, ω_x is a non-degenerate 2-form (bilnear symmetric) on $T_x(M)$:

$$T_x(M) \to T_x^*(M)$$

given by $u \mapsto \omega_x(u, -)$ is an isomorphism.

In fact the above two properties characterize symplectic forms:

Theorem 3.1.2 (Darbaux)**.** *Let M be a smooth manifold, and ω a differential 2-form on M such that*

1. $\mathrm{d}\omega = 0$*,*
2. *ω is non-degenerate*

Then ω is a symplectic structure on M.

We will prove this later.

Exercise 3.1.3. Show that a 2-form ω on M^{2n} is non-degenerate if and only if $\omega^{\wedge n}$ is never zero.

Now we list some examples of symplectic forms.

Example 3.1.4. $M = \mathbb{R}^{2n}, \omega = \omega_{\text{std}}$.

Example 3.1.5 (Cotangent bundle)**.** Let Q^n be a smooth n-manifold. Consider the cotangent space $T^*(Q)$, and the cotangent bundle of Q:

$$\begin{array}{c} T^*(Q) \\ \text{proj}\big\downarrow \\ Q \end{array}$$

Choose $q = (q_1, \ldots, q_n)$ to be local coordinates (chart) around some point in Q. Let $p = (p_1, \ldots, p_n)$ be the dual coordinates in $T^*(Q)$, i.e. if the basis e_i are the functionals (in T^*) such that

$$e_i\left(\frac{\partial}{\partial q_j}\right) = \delta_{ij}$$

Now define the *canonical form* on Q to be

$$\begin{aligned} \omega_{\text{can}} &\coloneqq \mathrm{d}p \wedge \mathrm{d}q = \mathrm{d}p_1 \wedge \mathrm{d}q_1 + \cdots + \mathrm{d}p_n \wedge \mathrm{d}q_n \\ &= \mathrm{d}(p_1 \mathrm{d}q_1 + \cdots + p_n \mathrm{d}q_n) \end{aligned}$$

Here is a coordinate-free description: Define a 1-form λ_{can} on $T^*(Q)$ as follows:

Consider the projection

$$\begin{array}{c} T^*(Q) \\ \text{proj}\big\downarrow \\ Q \end{array}$$

Let $q \in Q, p \in T_q^*(Q), v \in T_{(q,p)}(T^*(Q))$. Define

$$\lambda_{\text{can}}(v) \coloneqq \langle p, D\,\text{proj}(v)\rangle$$

where $D\,\text{proj}(v) \in T_q(Q)$. An exercise shows that in the local coordinates $q = (q_1, \ldots, q_n)$ and $p = (p_1, \ldots, p_n)$, λ_{can} looks like

$$\lambda_{\text{can}} = p_1 \mathrm{d}q_1 + \cdots + p_n \mathrm{d}q_n$$

and finally we define

$$\omega_{\text{can}} \coloneqq \mathrm{d}\lambda_{\text{can}}.$$

We call λ_{can} the *canonical Liouville form on $T^*(Q)$*.

Example 3.1.6. Take M an orientable surface, and ω any area form, i.e. locally:

$$\omega = f(x, y) \cdot \mathrm{d}x \wedge \mathrm{d}y, \quad f(x, y) \neq 0$$

3.2 Complex Projective Space as a Symplectic Manifold

A very important example of a symplectic maniflold that we will elaborate on is the complex projective space $\mathbb{C}P^n$ equipped with the symplectic form ω_{FS}. We first fix our definition of $\mathbb{C}P^n$:

Let $M = \mathbb{C}P^n$, where

$$\mathbb{C}P^n \coloneqq \left(\mathbb{C}^{n+1} \setminus \{0\}\right)/\!\sim$$

or the set of all complex lines in $\mathbb{C}^{n+1}$ that go through zero.

Proposition 3.2.1. *We have*

1. *$\mathbb{C}P^n$ is Hausdorff and compact.*

2. *$\mathbb{C}P^n$ can be endowed with a strcuture of a $2n$-dimensional manifold.*

Proof. These are well-known facts. □

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We show $\mathbb{C}P^n$ is compact by introducing the *Hopf fibration*: Consider $S^{2n+1} \subset \mathbb{C}^{n+1}$ as a subspace by

$$S^{2n+1} \cong \{(z_0, \dots, z_n) \in \mathbb{C}^{n+1} \mid |z_0|^2 + \cdots + |z_n|^2 = 1\}$$

The we can define the map

$$\begin{aligned} \pi \colon S^{2n+1} &\to \mathbb{C}P^n \\ (z_0, \dots, z_n) &\mapsto [z_0 : \cdots : z_n] \end{aligned}$$

by rescaling the homogeneous cooridnates we can ensure that π is surjective. It follows that $\mathbb{C}P^n$ is compact...why?

We can consider π as a fibration on $\mathbb{C}P^n$: The fiber

$$\pi^{-1}([z_0 : \cdots : z_n]) = \{(e^{it} z_0, \dots, e^{it} z_n) \mid t \in [0, 2\pi]\}$$

is homeomorphic to the circle S^1, and where we take the homogeneous cooridnates such that $|z_0|^2 + \cdots + |z_n|^2 = 1$ (possible by rescaling). Therefore in particular this fibration is locally trivial, i.e. it is in fact a fiber bundle. We call this the *Hopf fibration*.

Question. How can one visualize $\mathbb{C}P^n$?

Answer. $\mathbb{C}P^0$ is a point, while $\mathbb{C}P^1$ is homeomorphic to S^2. For higher dimensions, there is no hope.

3.2.1 (Singular) Homology of Complex Projective Spaces

Before we give the definition of the symplectic structure on $\mathbb{C}P^n$, we need some facts on the homology of the space. It is known that

$$H_k(\mathbb{C}P^n; \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & 0 \leq k \leq 2n \text{ and } k \text{ even} \\ 0 & k \text{ odd} \end{cases}$$

Fix $0 \leq r \leq n$ and consider

$$P_r := \{[z_0 : \cdots : z_n] \in \mathbb{C}P^n \mid z_{r+1} = \cdots = z_n = 0\}$$

then

$$P_r \cong \mathbb{C}P^r.$$

Now consider the map

$$j \colon \mathbb{C}P^r \to P_r \hookrightarrow \mathbb{C}P^n$$

defined by

$$[z_0 : \cdots : z_n] \mapsto [z_0 : \cdots : z_r : 0 : \cdots : 0]$$

then the induced map on homology

$$j_* \colon H_k(\mathbb{C}P^r) \to H_k(\mathbb{C}P^n) \quad \forall 0 \leq k \leq 2r$$

is an isomorphism. WHY?

3.2.2 The Symplectic Structure

Now we will define a symplectic form ω_{FS} on $\mathbb{C}P^n$ called the *Fubini-Study form.* In the chart U_0 (i.e. the one where the zeroth coordinate is non-zero) with local complex coordinates $(u_1, \ldots, u_n)$, where

$$u_1 = \frac{z_1}{z_0}, \ldots, u_n = \frac{z_n}{z_0}$$

we define

$$\omega_{\mathrm{FS}} = \frac{i}{2}\left(\frac{\sum \mathrm{d}u_k \wedge \mathrm{d}\bar{u}_k}{1+|u|^2} - \frac{(\sum \bar{u}_k \mathrm{d}u_k) \wedge (\sum u_k \mathrm{d}\bar{u}_k)}{(1+|u|^2)^2}\right)$$

which is horrible and we won't work with this. Instead we seek a more geometric definition.

Consider the Hopf fibreation

$$\pi\colon S^{2n+1} \to \mathbb{C}P^n$$

and inclusion

$$i\colon S^{2n+1} \hookrightarrow \mathbb{C}^{n+1}$$

Claim. there exists a 2-form ω_{FS} on $\mathbb{C}P^n$ such that

$$i^*\omega_{\mathrm{std}} = \pi^*\omega_{\mathrm{FS}}$$

Proof of claim. Let $x \in S^{2n+1} \hookrightarrow \mathbb{C}^{n+1}$. Then

$$T_x(\mathbb{C}^{n+1}) = \mathbb{R}x \oplus \mathbb{R}(ix) \oplus \xi_x$$

where

$$\xi_x := (\mathbb{R}x \oplus \mathbb{R}ix)^\perp$$

where the orthogonal complement we take with respect to the Euclidean metric. Since

$$T_x(S^{2n+1}) = (\mathbb{R}x)^\perp$$

it follows that $\xi_x \subset T_x(S^{2n+1})$ and ξ_x has dimension $2n$.

Recall we have the Hermitian product $\langle\cdot,\cdot\rangle$ on $\mathbb{C}^{n+1}$ and the scalar product $(\cdot,\cdot) = \mathrm{Re}\langle\cdot,\cdot\rangle$. We can write ξ_x as

$$\xi_x = \{v \in T_x(\mathbb{C}^{n+1}) \mid \langle v, x\rangle = 0\}$$

thus

$$T_x(S^{2n+1}) = \mathbb{R}ix \oplus \xi_x.$$

For every $t \in [0, 2\pi]$ consider the S^1-action on $\mathbb{C}^{n+1}$

$$\sigma_t\colon \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$$
$$x \mapsto e^{it}x$$

which leaves S^{2n+1} invariant

$$\sigma_t(S^{2n+1}) = S^{2n+1}$$

and

$$D\sigma_t(\xi_x) = \xi_{\sigma_t(x)}.$$

So σ_t preserves the splitting of the tangent space

$$T_x(S^{2n+1}) = \mathbb{R}ix \oplus \xi_x$$

See drawing.

$$D\pi_x\colon T_x(S^{2n+1}) \to T_{\pi(x)}(\mathbb{C}P^n)$$

sends $\mathbb{R}ix$ to 0. Indeed

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} e^{it}x = ix.$$

In addition, $D\pi_x$ sends ξ_x isomorphiscally to $T_{\pi(x)}(\mathbb{C}P^n)$. This follows from the local triviality of the Hopf fibration.

Note that $\sigma_t^*\omega_{\text{std}} = \omega_{\text{std}}$ and

$$(\sigma_t \circ i)^*\omega_{\text{std}} = i^*\omega_{\text{std}}$$

Also, $\pi \circ \sigma_t = \pi$ so $D\pi \circ D\sigma_t = D\pi$. Consequently,

$$\omega_{\text{std}}\Big|_{\xi_x} = \sigma_t^*\left(\omega_{\text{std}}\Big|_{\xi_{\sigma_t(x)}}\right)$$

Also,

$$\omega_{\text{std}}(ix, v) = 0 \quad \forall v \in \xi_x$$

Indeed since

$$\omega_{\text{std}}(ix, v) = -\operatorname{Im}\langle ix, v\rangle = 0 \quad \forall u \in \xi_x$$

Now

$$\omega_{\text{std}}(ix, ix) = 0, \quad \omega_{\text{std}}(ix, v) = 0 \quad \forall v \in \xi_x$$

so $\omega_{\text{std}}(ix, -) = 0$ on $T_x(S^{2n+1})$. Therefore there exits a 2-form $\Omega_{\pi(x)}$ on $T_{\pi(x)}(\mathbb{C}P^n)$ such that

$$\omega_{\text{std}}\Big|_{T_x(S^{2n+1})} = \pi^*\Omega_{\pi(x)}$$

The forms $\Omega_{\pi(x)}$ for $\pi(x) \in \mathbb{C}P^n$ give a global form ω_{FS} on $\mathbb{C}P^n$ such that

$$i^*\omega_{\text{std}} = \pi^*\omega_{\text{FS}}$$

□

Claim. Now that we have shown the existence of a form ω_{FS} on $\mathbb{C}P^n$, we claim that

1. ω_{FS} is closed,
2. ω_{FS} is non-degenerate.

Therefore ω_{FS} is a symplectic form on $\mathbb{C}P^n$.

Proof of claim. For closedness:

$$\mathrm{d}i^*\omega_{\text{std}} = \mathrm{d}\pi^*\omega_{\text{FS}}$$

so

$$i^*\mathrm{d}\omega_{\text{std}} = \pi^*\mathrm{d}\omega_{\text{FS}}.$$

Since we know that $\mathrm{d}\omega_{\text{std}} = 0$ and $D\pi$ is surjective, we conclude $\mathrm{d}\omega_{\text{FS}} = 0$.

For non-degeneracy: Let $0 \neq v \in T_z(\mathbb{C}P^n)$. Pick any $x \in \pi^{-1}(z)$ in the fiber, and pick $v' \in \xi_x$ such that $D\pi(v') = v$ (by surjectivity). Note that $iv \in \xi_x$ because

$$\langle iv', x\rangle = i\langle v', x\rangle = 0$$

Now

$$\begin{aligned}
\omega_{\text{FS}}(v, D\pi(iv')) &= \omega_{\text{FS}}(D\pi(v'), D\pi(iv')) \\
&= \pi^*\omega_{\text{FS}}(v', iv') \\
&= \omega_{\text{std}}(v', iv') \\
&= \operatorname{Im}\langle v', iv'\rangle \\
&= -\operatorname{Im}(-i\langle v', v'\rangle) \\
&= |v'|^2 > 0
\end{aligned}$$

So ω_{FS} is non-degenerate. □

Now we present some properties of the Fubini-Study form.

Let $A \in \mathrm{GL}(n+1, \mathbb{C})$. Then A induces a diffeomorphism

$$\begin{aligned}\bar{A} \colon \mathbb{C}P^n &\to \mathbb{C}P^n \\ [z_0 : \cdots : z_n] &\mapsto [A \cdot \begin{pmatrix} z_0 \\ \vdots \\ z_n \end{pmatrix}]\end{aligned}$$

Proposition 3.2.2. *Let $T \in \mathrm{U}(n+1, \mathbb{C})$. Then $\bar{T}^* \omega_{FS} = \omega_{FS}$.*

Proof. Key points to consider are the following:

- $T(S^{2n+1}) = S^{2n+1}$
- $T^* \omega_{\mathrm{std}} = \omega_{\mathrm{std}}$
- $\langle Tu, Tv \rangle = \langle u, v \rangle$
- $i^* \omega_{\mathrm{std}} = \pi^* \omega_{\mathrm{FS}}$

$\square$

04.03.2020

Proposition 3.2.3. *Let $S \subset \mathbb{C}P^n$ be a projective line $\mathbb{C}P^1$ with its standard orientation. Then*

$$\int_S \omega_{FS} = \pi$$

Proof. Parametrize S, W.O.L.G by

$$\begin{aligned}\iota \colon \mathbb{C}P^1 &\hookrightarrow \mathbb{C}P^n \\ [z_0 : z_1] &\mapsto [z_0 : z_1 : 0 : \cdots : 0]\end{aligned}$$

We want to prove

$$\int_{\mathbb{C}P^1} \iota^* \omega_{\mathrm{FS}} = \pi.$$

To that end, it is enough to prove

$$\int_{\mathbb{C}P^1 \setminus \{[0:1]\}} \iota^* \omega_{\mathrm{FS}} = \pi.$$

We denote by $\iota' \colon \mathbb{C}P^1 \setminus \{[0:1]\} \hookrightarrow \mathbb{C}P^n$ the map ι restricted to $\mathbb{C}P^1 \setminus \{[0:1]\}$. Let us also introduce the map

$$\begin{aligned}j \colon \mathbb{C} &\to \mathbb{C}P^1 \\ z &\mapsto [1 : z]\end{aligned}$$

Consider the composition

$$\mathbb{C} \xrightarrow{j} \mathbb{C}P^1 \setminus \{[0:1]\} \xrightarrow{\iota'} \mathbb{C}P^n$$

Then it is enough to prove

$$\int_{\mathbb{C}} j^* \iota'^* \omega_{\mathrm{FS}} = \pi.$$

The point is that $\iota' \circ j \colon \mathbb{C} \to \mathbb{C}P^n$ lifts

$$\begin{array}{ccc} & & S^{2n+1} \\ & \overset{s}{\nearrow} & \big\downarrow \pi \\ \mathbb{C} & \xrightarrow[\iota' \circ j]{} & \mathbb{C}P^n \end{array}$$

obtaining

$$s(z) := \left(\frac{1}{(1+|z|^2)^{1/2}}, \frac{z}{(1+|z|^2)^{1/2}} \right)$$

so

$$\int_{\mathbb{C}} j^* \iota'^* \omega_{\mathrm{FS}} = \int_{\mathbb{C}} s^* \pi^* \omega_{\mathrm{FS}} = \int_{\mathbb{C}} s^* \left(\omega_{\mathrm{std}} \Big|_{S^{2n+1}} \right). \tag{3.1}$$

So the goal now is to calculate the right-hand-side of this. Parametrize $\mathbb{C}$ by the interiot of the disk $\operatorname{Int} D = \{ w \in \mathbb{C} \mid |w| < 1 \}$. Let $h \colon [0,1) \to [0,\infty)$ be a diffeomorphism such that

1. $h(r) = r$ for r near 0,
2. h is strictly increasing,
3. $h(r) \xrightarrow{r \to 1^-} \infty$

Then define

$$\begin{aligned} \kappa \colon \operatorname{Int} D &\to \mathbb{C} \\ w &\mapsto h(|w|) \cdot w \end{aligned}$$

Then we can write the right hand side of (3.1) as

$$\int_{\mathbb{C}} s^* \left(\omega_{\mathrm{std}} \Big|_{S^{2n+1}} \right) = \int_{\operatorname{Int} D} \underbrace{\kappa^* s^*}_{(s \circ \kappa)^*} \left(\omega_{\mathrm{std}} \Big|_{S^{2n+1}} \right)$$

Note: $s \circ \kappa \colon S^{2n+1} \subset \mathbb{C}^{n+1}$ extends smoothly to a map

$$\nu \colon D \to S^{2n+1} \subset \mathbb{C}^{n+1}$$

defined by

$$\nu(w) := \begin{cases} \left(\frac{1}{(1+h(|w|)^2 \cdot |w|^2)^{1/2}}, \frac{h(|w|) \cdot w}{(1+h(|w|)^2 \cdot |w|^2)^{1/2}}, 0, \ldots, 0 \right) & |w| < 1 \\ (0, w, 0, \ldots, 0) & |w| = 1 \end{cases}$$

Clearly:

$$\int_{\operatorname{Int} D} (s \circ \kappa)^* \omega_{\mathrm{std}} = \int_D \nu^* \omega_{\mathrm{std}}$$

There are infinitely many primitives λ for ω_{std} (i.e. for which $\omega_{\mathrm{std}} = \mathrm{d}\lambda$). We will use

$$\lambda_0 := \frac{1}{2} \sum_{k=0}^{n} x_k \mathrm{d}y_k - y_k \mathrm{d}x_k.$$

Parametrize $\nu\big|_{\partial D} \colon \partial D \to \mathbb{C}^{n+1}$ by $\gamma(t) = (0, e^{it}, 0, 0, \ldots, 0), t \in [0, 2\pi]$. So using Stokes' Theorem

$$\begin{aligned} \int_D \nu^* \omega_{\mathrm{std}} &= \int_{\partial D} \nu^* \lambda_0 \\ &= \int_0^{2\pi} \lambda_{0_{(\gamma(t))}} (\dot{\gamma}(t)) \mathrm{d}t \end{aligned}$$

where

$$\gamma(t) = \begin{cases} x_0 = 0, y_0 = 0 \\ x_1 = \cos t, y_1 = \sin t \\ x_j = y_j = 0 \forall j \geq 2 \end{cases}$$

and thus

$$\dot{\gamma}(t) = -\sin t \frac{\partial}{\partial x_1} + \cos t \frac{\partial}{\partial y_1} \lambda_0$$

therefore

$$\lambda_{0_{(\gamma(t))}}(\dot{\gamma}(t)) = \frac{1}{2}(\cos^2 t - \sin t(-\sin t)) = \frac{1}{2}.$$

which gives us

$$\int_0^{2\pi} \lambda_{0_{(\gamma(t))}}(\dot{\gamma}(t)) \mathrm{d}t = \pi$$

as desired. □

Exercise 3.2.4.

3.2.3 Cohomology of Complex Projective Spaces

Consider the injection

$$\mathbb{Z} \hookrightarrow \mathbb{R}$$

whose induced map on cohomology rings is

$$e \colon H^*(\mathbb{C}P^n, \mathbb{Z}) \to H^*(\mathbb{C}P^n, \mathbb{R}).$$

Exercise 3.2.5. e is injective. The class

$$\left[\frac{1}{\pi}\omega_{\mathrm{FS}}\right] \in H^2_{\mathrm{d}R}(\mathbb{C}P^2) \cong H^2(\mathbb{C}P^2, \mathbb{R})$$

is in the image of e. This defines a unique class

$$h \in H^2(\mathbb{C}P^n, \mathbb{Z})$$

Claim. h generates $H^*(\mathbb{C}P^n, \mathbb{Z})$ with respect to the cup product.

Claim. $h = \mathrm{PD}[P_{n-1}]$ (Poincaré dual). Where $[P_{n-1}] \in H_{2n-2}(\mathbb{C}P^n, \mathbb{Z})$ is the homology class of $\mathbb{C}P^{n-1} \subset \mathbb{C}P^n$ where we view $\mathbb{C}P^{n-1} = \{[z_0 : \cdots : z_n] \mid z_0 = 0\} \subset \mathbb{C}P^n$

Question. How to write ω_{FS} in local coordinates?

Answer. Consider the chart

$$\phi \colon \mathbb{C}^n \to \mathbb{C}P^n$$
$$(z_1, \ldots, z_n) \mapsto [1, z_1 : \cdots : z_n]$$

ϕ admits a lift

$$\begin{array}{ccc} & & S^{2n+1} \\ & \overset{\tilde{\phi}}{\nearrow} & \downarrow \pi \\ \mathbb{C}^n & \xrightarrow{\phi} & \mathbb{C}P^n \end{array}$$

where

$$\tilde{\phi}(z_1, \ldots, z_n) = \left(\frac{1}{(1 + |z|^2)^{1/2}}, \ldots, \frac{z_n}{} \right)$$

$$\phi^* \omega_{\mathrm{FS}} = \tilde{\phi}^* \pi^* \omega_{\mathrm{FS}} = \tilde{\phi}^* \omega_{\mathrm{std}} = \ldots \textit{explicitly???} = \frac{i}{2} \partial \bar{\partial} \log\Big(1 + |\underline{z}|^2\Big)$$

Proposition 3.2.6. *Let (M^{2n}, ω) a closed positive dimensional (i.e. not a point) symplectic manifold. Then $[\omega] \neq 0 \in H^*_{\mathrm{d}R}(M)$.*

Proof. Assume by contradiction that $\omega = \mathrm{d}\lambda$. Then wedge ω n-times we have

$$\omega^{\wedge n} = (\mathrm{d}\lambda)^{\wedge n} = \mathrm{d}\Big(\lambda \wedge \omega^{\wedge(n-1)}\Big)$$

then by Stokes

$$\int_M \omega^{\wedge n} = \int_{\partial M} \lambda \wedge \omega^{\wedge(n-1)} = 0$$

this is impossible because $\omega^{\wedge n}$ is a volume form. □

05.03.2020

If we take $V \subset \mathbb{C}^N$ a complex algebriac manifold, meaning that V is an affine algebraic set. Then $\omega_{\mathrm{std}}\big|_V$ is non-degenerate. So $(V, \omega_{\mathrm{std}}\big|_V)$ is a symplectic manifold. We can take holomorphic instead of algebriac it will still work.

Example 3.2.7. Take

$$V = \{(z_1, \ldots, z_N) \in \mathbb{C}^N \mid z_1^2 + \cdots + z_N^2 = 0\}$$

In a similar way, if $X \subset \mathbb{C}P^n$ a complex algebraic projective manifold, i.e. a projective algebraic set. Then $(X, \omega_{\mathrm{FS}}\big|_X)$ is symplectic.

We will not prove these at the moment. Notice that these already give a huge number of examples of symplectic manifolds. Also notice that in the case of complex algebraic manifold, it can never compact; while the case for projective manifolds is always compact.

Now take $(M_1, \omega_1), (M_2, \omega_2)$ to be symplectic manifolds. Then $(M_1 \times M_2, \omega_1 \oplus \omega_2)$ is a symplectic manifold; where $\omega_1 \oplus \omega_2 := p_1^*\omega_1 + p_2^*\omega_2$.

It is not obvious how to construct symplectic manifolds. Most of the methods we know from topology to construct new spaces from old will not give us symplectic manifolds.

3.3 Interlude: Flows on Manifolds

Let I be an interval that contains 0, usually we take $I = [0, 1]$. Let M be a smooth manifold.

A *flow on M* is a one-parameter family $\{\phi_t\}_{t\in I}$ of diffeomorphisms $\phi_t \colon M \to M$, and for which $\phi_0 = \mathrm{Id}$. A flow will give us a time-dependent vector field on M denoted by $\{X_t\}_{t\in I}$. The relation between the two is

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi_t(x) = X_t(\phi_t(x)).$$

More precisely, given a flow $\{\phi_t\}_{t\in I}$, define

$$X_t(y) := \frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=t} \phi_s(\phi_t^{-1}(y))$$

Conversely, if given $\{X_t\}_{t\in I}$ a time-dependent vector field, we can define a flow by

$$\begin{cases} \dot{\phi}_t(x) = X_t(\phi_t(x)) \\ \phi_0(x) = x \end{cases}$$

which is a system of O.D.E.'s.

If $X_t \equiv X$ for all t, i.e. an *autonomous or time independent vector field.* Then its flow satisfies

$$\phi_{t+s}(x) = \phi_t \circ \phi_s(x) = \phi_s \circ \phi_t(x)$$

for all $t, s \in I$, and the map

$$\begin{aligned} \mathbb{R} &\to \mathrm{Diff}(M) \\ t &\mapsto \phi_t \end{aligned}$$

where $\mathrm{Diff}(M)$ is the group of diffeomorphisms from M to itself, is a group homomorphism.

Remark. In a non-autonomous vector field, the trajectory of the flow can intersect itself at different times.

But how do we even know that flows actually exist?

Suppose M is a non-compact manifold without boundary, and X a vector field on M, and $x_0 \in M$. Usually, there exists a real number $T_0(x)$ such that the trajectory of X, call it $\phi_t(x_0)$ going through x_0 at $t = 0$ exists only for $|t| < T(x_0)$. If $\inf_{x_0 \in M} T(x_0) = 0$ we do not have a flow.

Indeed, below is an example of an autonomous vector field on $\mathbb{R}$, for which there does not exist a corresponding flow.

Example 3.3.1. Let $M = \mathbb{R}$ and $X(x) = x^2 \cdot \frac{\partial}{\partial x}$. To calculate the flow we solve the system of O.D.E.'s:

$$\begin{cases} \dot{x}(t) = x(t)^2 \\ x(0) = x_0 \end{cases}$$

The unique solution is

$$x(t) = \frac{x_0}{1 - x_0 \cdot t}$$

This is defined say for $x_0 > 0$ and on $t < \frac{1}{x_0}$.

What are some conditions to ensure that X_t define a flow?

- If M is closed, the ϕ_t is defined for all $t \in I$.
- Consider the support
$$\operatorname{supp}(\{X\}_{t \in I}) = \bigcup_{t \in I} \overline{\{x \in M \mid X_t(x) \neq 0\}}.$$
If $\operatorname{supp}(\{X_t\}_{t \in I})$ is compact, then the flow is well-defined for all t.
- Also consider the support
$$\operatorname{supp}\{\phi_t\}_{t \in I} = \bigcup_{t \in I} \overline{\{x \in M \mid \phi_t(x) \neq x\}}$$
- Suppose X_t satisfies the following: There exists a point $x_0 \in M$ such that $X_t(x_0) = 0$ for all $t \in I$, and that I is compact. Then the theory of O.D.E.'s tells us that there exists a neighborhood U of x_0 such that the "flow" $\phi_t(x)$ of X_t is defined for all $x \in U$ and all time $t \in I$. This gives us embeddings $\phi_t \colon U \hookrightarrow M$ for all $t \in I$.

3.4 Symplectic Diffeomorphisms a.k.a. Symplectomorphisms

Definition 3.4.1. A diffeomorphism

$$\phi \colon (M, \omega) \to (N, \omega')$$

is called a *symplectic diffeomorphism or symplectomorphism* if

$$\phi^* \omega' = \omega$$

How do we construct such a morphism?

3.4.1 Hamiltonian diffeomorphisms

Let (M, ω) be a symplectic manifold. Let

$$H \colon M \times I \to \mathbb{R}, \quad I = [0, 1]$$

be a smooth function. Denote $H_t(x) := H(x, t)$. Define a vector field (time dependent) X_t^H as follows. Consider the 1-forms $\mathrm{d}H_t$, where t ranges over I. Define X_t^H by the following equation:

$$\omega(X_t^H(x), -) = -\mathrm{d}H_{t(x)}(-)$$

Denote by $\{\phi_t^H\}_{t \in I} \colon M \to M$ the flow of X_t^H, if it exists.

Proposition 3.4.2. $(\phi_t^H)^*\omega = \omega$ *for all* t*. In particular,* $(\phi_t^H)^*\omega^{\wedge n} = \omega^{\wedge n}$*, so* ϕ_t^H *also preserves volmume.*

Definition 3.4.3 (Lie derivative)**.** Suppose Y is a time independent vector field on M, and β a k-form. Then

$$L_Y(\beta) \coloneqq \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} (\Psi_t^*\beta)$$

where Ψ_t is the flow of Y. This thing is a k-form.

The *homotopy formula* is the following

$$L_Y\beta = \mathrm{d}iy\beta + iy\mathrm{d}\beta$$

Recall the *fisherman formula*: If Y_t is a time dependent vector field on a manifold W with flow ϕ_t and α_t is a smooth family of k-forms depending on t. Then

$$\frac{\mathrm{d}}{\mathrm{d}t}(\phi_t^*\alpha_t) = \phi_t^*(L_{Y_t}\alpha_t + \dot{\alpha}_t)$$

Proof of Proposition.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(\phi_t^H)^* &= (\phi_t^H)^*(L_{X_t^H}\omega) \\
&= (\phi_t^H)^*(\mathrm{d}\underbrace{i_{X_t^H}\omega}_{-\mathrm{d}H_t} + i_{X_t^H}\underbrace{\mathrm{d}\omega}_{=0}) \\
&= (\phi_t^H)^*(-\mathrm{d}\mathrm{d}H_t \\
&= 0
\end{aligned}$$

So

$$(\phi_t^H)^*\omega = \underbrace{(\phi_0^H)^*}_{=\mathrm{Id}}\omega = \omega$$

□

We call ϕ_t^H the *Hamiltonian flow of* H. A diffeomorphism $\phi\colon M \to M$ is called *Hamiltoninan* if ther eexists H such that $\phi = \phi_1^H$.

Exercise 3.4.4. Take a Darbaux chart $\phi\colon U \subset \mathbb{R}^{2n} \to M$ such taht $\phi^*\omega = \omega_{\mathrm{std}}$. Take $H\colon U \times I \to \mathbb{R}$. The O.D.E.'s for ϕ_t^H are

$$\begin{cases} \dot{p}(t) = -\frac{\partial H}{\partial q}(p(t), q(t), t) \\ \dot{q}(t) = \frac{\partial H}{\partial p}(p(t), q(t), t) \end{cases}$$

$$X_t^H = \underbrace{-\frac{\partial H}{\partial q}\frac{\partial}{\partial p}}_{\sum_{k=1}^n \frac{\partial H}{\partial q_k}\frac{\partial}{\partial p_k}} + \frac{\partial H}{\partial p}\frac{\partial}{\partial q}$$

Appendix A

Crash Course on Smooth Manifolds

Appendix B

Crash Course on Homology and Cohomology