due 10/16 2:00pm

Mock Midterm exam 2

90 minutes

Name:

Section:

Instructions.

1. No quiz this week, but there will be a quiz next week after the midterm.

Question 1 Consider a curve $\vec{r}(t) = (2t, 1-3t, 5+4t)$ for $t \geq 0$.

(1) Calculate the length of the curve with $0 \leq t \leq t_0$.

(2) Find the arc-length parameterization $\vec{r}(s)$ where s is the arc-length parameter.

(3) Calculate $\frac{\mathrm{d}\vec{r}}{\mathrm{d}s}$ and $\frac{\mathrm{d}^2\vec{r}}{\mathrm{d}s^2}$.

(4) Calculate the curvature $\kappa(s)$ of the curve using the arc-length parameter.

Solution:

1. We use the formula for arc length:

$$L = \int_0^{t_0} |\vec{r}'(t)| \mathrm{d}t = \int_0^{t_0} |\langle 2, -3, 4\rangle| \mathrm{d}t = \int_0^{t_0} \sqrt{29} \mathrm{d}t = \sqrt{29} t_0.$$

2. First we employ the arc length function:

$$s(t) = \int_0^t |\vec{r}'(u)| \mathrm{d}u = \sqrt{29} t.$$

Thus $s(t) = \sqrt{29}t$, so solving for t we have

$$t(s) = \frac{s}{\sqrt{29}}.$$

Then plugging into $\vec{r}$ we have

$$\vec{r}(s) = \vec{r}(t(s)) = \left\langle \frac{2s}{\sqrt{29}}, 1 - \frac{3s}{\sqrt{29}}, 5 + \frac{4s}{\sqrt{29}} \right\rangle.$$

3. Using the previous result, we can obtain

$$r'(s) = \frac{\mathrm{d}r}{\mathrm{d}s} = \frac{\mathrm{d}}{\mathrm{d}s} \left\langle \frac{2s}{\sqrt{29}}, 1 - \frac{3s}{\sqrt{29}}, 5 + \frac{4s}{\sqrt{29}} \right\rangle = \left\langle \frac{2}{\sqrt{29}}, \frac{3}{\sqrt{29}}, \frac{4}{\sqrt{29}} \right\rangle.$$

and then trivially,

$$\frac{\mathrm{d}^2 r}{\mathrm{d}s^2} = \langle 0, 0, 0\rangle.$$

4. We use the formula

$$\kappa(s) = \left| \frac{\mathrm{d}T}{\mathrm{d}s} \right|.$$

But actually since $T(s) = \frac{r'(s)}{|r'(s)|}$ and we know that $r'(s)$ only has constants in its coordinates, the derivative of T must be the zero vector:

$$\kappa(s) = \left| \frac{\mathrm{d}T}{\mathrm{d}s} \right| = 0$$

. If you want you can actually check this: First we need

$$|r'(s)| = \sqrt{(2/\sqrt{29})^2 + (1 - 3/\sqrt{29})^2 + (4/\sqrt{29})^2} = 3 \quad \text{this is a tedius calculation}$$

Then

$$T(s) = \frac{r'(s)}{|r'(s)|} = \frac{1}{3}\left\langle \frac{2}{\sqrt{29}}, \frac{3}{\sqrt{29}}, \frac{4}{\sqrt{29}} \right\rangle$$

only has constants in all its coordinates so its derivative must be zero, so trivially

$$\kappa(s) = \left| \frac{\mathrm{d}T}{\mathrm{d}s} \right| = 0.$$

Question 2 Consider the function $f(x, y, z) = \sin(3x + yz)$ and the unit vector $\vec{u} = (\frac{1}{3}, \frac{2}{3}, 1)$. Find:

1. The partial derivative of f with respect to x.
2. The partial derivative of f with respect to y.
3. The partial derivative of f with respect to z.
4. The gradient of f.
5. The directional derivative of f along/in the direction of $\vec{u}$.

Solution:

1. $$\frac{\partial f}{\partial x} = 3\cos(3x + yz).$$
2. $$\frac{\partial f}{\partial y} = z\cos(3x + yz).$$
3. $$\frac{\partial f}{\partial z} = y\cos(3x + yz).$$
4. (this will not be covered in the exam)
$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle = \langle 3\cos(3x + yz), z\cos(3x + yz), y\cos(3x + yz) \rangle .$$
5. (this will not be covered in the exam)
$$\begin{aligned} D_u f &= \nabla f \cdot u \\ &= \langle 3\cos(3x + yz), z\cos(3x + yz), y\cos(3x + yz) \rangle \cdot \left\langle \frac{1}{3}, \frac{2}{3}, 1 \right\rangle \\ &= \cos(3x + yz) + \frac{2z}{3}\cos(3x + yz) + y\cos(3x + yz). \end{aligned}$$

Question 3 Consider the function $f(x,y) = \ln\big(x^2 + 4y^2\big)$.

1. What is the domain of definition of $f(x,y)$?
2. What is the range of the function $f(x,y)$?

Solution: For the domain, the only thing we need to worry about is that ln cannot take inputs that are non-positive. However, since $x^2 + 4y^2$ is always non-negative, we only need to worry if $x^2 + 4y^2 = 0$. Thus the domain is

$$\{(x,y) \in \mathbb{R}^2 \mid x^2 + 4y^2 \neq 0\}.$$

Since the range of ln is all of $\mathbb{R}$, the range of $\ln\big(x^2 + 4y^2\big)$ is all of $\mathbb{R}$ as well.

Question 4

1. Write down a vector valued function $\vec{r}(t)$ that corresponds to a circle on the plane centered at 0 with radius 1.
2. Compute the curvature of the circle.

Solution:

1. You have seen in class that the function is

$$\vec{r}(t) = \langle \cos t, \sin t \rangle.$$

2. First we compute the derivative

$$\vec{r}'(t) = \langle -\sin t, \cos t \rangle.$$

Then normalizing

$$T(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|} = \frac{\langle -\sin t, \cos t \rangle}{\sqrt{\sin^2 t + \cos^2 t}} = \langle -\sin t, \cos t \rangle.$$

Thus

$$T'(t) = \langle -\cos t, -\sin t \rangle.$$

So finally

$$\kappa(t) = \frac{|T'(t)|}{|\vec{r}'(t)|} = \frac{1}{1} = 1.$$

Question 4 State what it means for a function $f(x,y)$ to be continuous at a point $(a,b) \in \mathbb{R}^2$.
Solution: $f(x,y)$ is continuous at (a,b) if

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b).$$

Question 5 [**points**] Find the limit, if it exists, or show that the limit does not exist.

$$(\mathbf{1}) \lim_{(x,y)\mapsto(0,0)} \frac{\sin(xy)}{xy}.$$

$$(\mathbf{2}) \lim_{(x,y)\mapsto(0,0)} \frac{y^4}{x^4+3y^4}.$$

Solution:

1. Let $F(x,y) = \frac{\sin(xy)}{xy}$. Then we can write F as a composition by defining

$$g(x,y) = xy,$$

and

$$f(t) = \frac{\sin(t)}{t}.$$

Then since

$$\lim_{(x,y)\to(a,b)} g(x,y) = \lim_{(x,y)\to(a,b)} xy = 0,$$

and

$$\lim_{t\to 0} f(t) = \lim_{t\to 0} \frac{\sin t}{t} = 1,$$

Using the Composite Function Theorem, we have

$$\lim_{(x,y)\mapsto(0,0)} \frac{\sin(xy)}{xy} = \lim_{(x,y)\to(0,0)} F(x,y) = \lim_{(x,y)\to(0,0)} f\circ g(x,y) = \lim_{t\to 0} f(t) = 1.$$

(For details see my Notes on Limits, but for the exam writing something like this is all that is necessary.)

2. The limit does not exists because if we take the path $y = mx$, we get

$$\lim_{(x,y)\to(0,0)} \frac{(mx)^4}{x^4+3(mx)^4} = \lim_{(x,y)\to(0,0)} \frac{m^4}{3m^4} = \frac{1}{3}.$$

But if we take the path $x = my$, we get

$$\lim_{(x,y)\to(0,0)} \frac{y^4}{(my)^4+3y^4} = \lim_{(x,y)\to(0,0)} \frac{1}{m^4+3}.$$

So if we choose some $m \neq 0$ then we will have different values for the two limits.

Question 6

(1) Find the linear approximation of $f(x,y) = 3y^2 - 2x^2 + x$ at the point $(x,y) = (1,1)$. Simplify your answer to the form $L(x,y) = Ax + By + C$.

Solution: First let's plug in $(1,1)$ into the function to get $f(1,1) = 2 = z_0$. So our point is $(x_0, y_0, z_0) = (1, 1, 2)$.

Now we find the equation of the tangent plane at (x_0, y_0, z_0):

$$z - z_0 = \frac{\partial f(x_0, y_0)}{\partial x}(x - x_0) + \frac{\partial f(x_0, y_0)}{\partial y}(y - y_0)$$

Since

$$\frac{\partial f}{\partial x} = -4x + 1$$

and

$$\frac{\partial f}{\partial y} = 6y$$

we have

$$z - 2 = (-4(1) + 1)(x - 1) + (6(1))(y - 1)$$

which is equivalent to

$$3x + 6y - z = 7.$$

Solving for z we have

$$z = 3x + 6y - 7.$$

So the linear approximation is

$$L(x, y) = 3x + 6y - 7.$$

Question 7

Review implicit differentiation and make sure you know how to do it, specifically be able to use Equation 6 and Equation 7 in Section 11.5.

Solution: Read pages 785 and 786 in the textbook.