Notes on Limits

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1 Preliminaries

Hello everyone. Thanks for taking the time to read these notes I've written. I must apologize first that this whole limit business must be pretty confusing to many of you, since you did not see much of it in lecture or homework, and you are somehow expected to know how to do them. I also want to apologize to those who thought some of my explanations in recitation were not very clear, I have to confess it has been a very long time since I've had to evaluate a limit. Plus, evaluating limits is one of those things that once you have successfully escaped from, you just never want to think about at all again. This has to do with the fact that knowing how to evaluate a limit is actually *not very important*. It really is not the focus of this course at all because it really just comes down to being really good at doing tricks to re-write functions, you know, the kind of skill that is useful in math competitions but completely useless in understanding the actual mathematics behind these things. This is also why **there will only be two fairly easy limit problems on the upcoming midterm**. So I really don't want you guys to spend to much time on this because this will really distract you from the really important stuff like derivatives and integrals.

What *is* important is to take way from all this as you continue with this course is that for a multivariable function, the limit must be the same no matter which path you take to approach it, and this is how you show a limit does not exist. Another important thing to know is that at a continuous point, the limit is just the value of the function there. These are basically the two most important things to know. I suppose also you should know how to evaluate limits in the single variable case, but most of what you will encounter are typical continuous functions such as polynomials, exponentials, etc. Or something you can solve with L'Hospital's Rule.

Again I want to emphasize that knowing exactly how to evaluate a limit is completely fucking useless in understanding multivariable calculus. Knowing what a limit is important, and knowing how it behaves is important, but knowing these tricks to actually *evaluating* them is a bullshit skill that doesn't show any deeper understanding of the mathematics. So please, if you are turned off by this just know that this is not the kind of stuff that mathematicians do at all, none of us care about this skill. And I am so sorry that jackasses like James Stewart must go out there and write a calculus textbook that makes it seem like this is important, and thus these calculus curriculum we have in our education across this country must adhere to this type of degeneracy.

I promise you that the limit problems on the exams each will not take you more than 5 minutes to do, and won't include anything not explained in these notes.

2 Continuous Functions

First I will discuss continuous functions, the nicest types of functions.

Definition 2.1. Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a real-valued function. Let $x = (x_1, \dots, x_n) \in \mathbb{R}^n$. We say that f is *continuous at* x if for any $\varepsilon > 0$, there exists a $\delta > 0$ such that for any $y = (y_1, \dots, y_n) \in \mathbb{R}^n$, we have

$$|x - y| < \delta \quad \text{implies} \quad |f(x) - f(y)| < \varepsilon.$$

Equivalently,

A function $f\colon \mathbb{R}^n \to \mathbb{R}$ is called continuous at $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ if

$$\lim_{(y_1,\dots,y_n)\to(x_1,\dots,x_n)} f(y_1, \dots, y_n) = f(x_1, \dots, x_n).$$

We will mostly be dealing with two-variable functions so this is the following:

Definition 2.2. A function f of two variables is called *continuous at (a,b)* if

$$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b).$$

We say that a function is *continuous* if it is continuous at every point.

Most of the single-variable functions you have learned in previous courses are continuous under most circumstances. Functions such as polynomials, most trig functions, exponentials, and logarithms are continuous if the appropriate domain is specified. You should spend some time getting comfortable with knowing where these functions are continuous, and where they are not. Most importantly, *a function cannot be continuous at a point at which it is not defined.*

Proposition 2.3. *The composition of continuous functions is continuous. That is, if* $f\colon \mathbb{R}^n \to \mathbb{R}$ *and* $g\colon \mathbb{R} \to \mathbb{R}$ *are continuous real-valued functions, then* $g \circ f\colon \mathbb{R}^n \to \mathbb{R}$ *is continuous.*

Although this result holds true for more general functions $f\colon \mathbb{R}^n \to \mathbb{R}^m$, and $g\colon \mathbb{R}^m \to \mathbb{R}^k$, the version given above is really what we will need.

Example 2.4. Let $f\colon \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x + 3$, and $g\colon \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = e^x$. It is clear that f is continuous as it is a polynomial function, and g is continuous it is an exponential function that is defined everywhere. Thus by the Proposition above, the composition

$$\begin{aligned} g \circ f\colon \mathbb{R} &\to \mathbb{R} \\ x &\mapsto e^{(x+3)} \end{aligned}$$

is continuous.

Now an example with a two-variable function:

Example 2.5. Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(a, b) = a + b$, and $g\colon \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = x^2$. Then it is clear that both functions are continuous since they are both polynomial functions. Thus by the above proposition, the composition

$$\begin{aligned} g \circ f\colon \mathbb{R}^2 &\to \mathbb{R} \\ (a, b) &\mapsto (a + b)^2 \end{aligned}$$

is a continuous function.

Now when evaluating limits, as we know on a continuous function you just use the value of the function. Actually, since we also know that composition of continuous functions is continuous, we also have:

Corollary 2.6. *Suppose* $f\colon \mathbb{R}^n \to \mathbb{R}$ *and* $g\colon \mathbb{R} \to \mathbb{R}$ *are continuous functions, then for any* $a \in \mathbb{R}^n$*,*

$$\lim_{x\to a}(g \circ f)(x) = g(f(a))$$

Proof. Using the definition of continuity that says that the limit at any point equals the value of the function at that point, we can write

$$\begin{aligned} \lim_{x\to a}(g \circ f)(x) &= \lim_{x\to a} g(f(x)) \\ &= g(f(a)) \quad \text{because } g \circ f \text{ is continuous} \end{aligned}$$

□

Example 2.7. Evaluate the limit

$$\lim_{(x,y)\to(0,0)} \sin(xy).$$

We can write the function $\sin(xy)$ as the composition of two continuous functions $f(x, y) = xy$, and $g(t) = \sin(t)$, so the limit becomes

$$\lim_{(x,y)\to(0,0)} \sin(xy) = \lim_{(x,y)\to(0,0)} (g \circ f)(x) = g(f(0, 0)) = \sin(0) = 0.$$

But in fact, as long as the point at which we are actually evaluating has nice continuity properties, we can pretty much do the same thing.

Proposition 2.8 (I think this is called something like "Composite Function Theorem", I'm not sure though)**.** *Let $f\colon \mathbb{R}^n \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$ be real-valued functions. If $\lim_{x\to a} f(x) = L$, and*

$$\lim_{t\to L} g(t)$$

exists, then

$$\lim_{x\to a} g(f(x)) = \lim_{t\to L} g(t)$$

Example 2.9. Evaluate the limit

$$\lim_{(x,y)\to(0,0)} \frac{\sin(xy)}{(xy)}$$

Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be defined by $f(x,y) = xy$, and $g\colon \mathbb{R} \to \mathbb{R}$ be defined by $g(t) = \frac{\sin(t)}{t}$. Then the limit becomes the limit of the composition $g \circ f$ at $(0,0) \in \mathbb{R}^2$:

$$\lim_{(x,y)\to(0,0)} \frac{\sin(xy)}{(xy)} = \lim_{(x,y)\to(0,0)} (g\circ f)(x,y).$$

Now we know that

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} xy = 0.$$

by continuity of f at $(0,0)$, since this is a polynomial. Let us set $L = 0$. Then the limit

$$\lim_{t\to L} g(t) = \lim_{t\to 0} \frac{\sin(t)}{t} = 1$$

exists and takes the value 1. Thus by the Proposition, we have

$$\lim_{(x,y)\to(0,0)} (g\circ f)(x,y) = \lim_{(x,y)\to(0,0)} \frac{\sin(xy)}{xy} = \lim_{t\to 0} \frac{\sin(t)}{t} = 1.$$

By the way, one of the important reasons why this works is due to the fact that the limit

$$\lim_{t\to 0} \frac{\sin(t)}{t} = 1$$

actually exists, which is equivalent to the fact that $\frac{\sin t}{t}$ has a removable discontinuity at 0. And if you remember from Calc. 1 or Calc. 2, if a function f only has removable discontinuities, then they can be "removed" to create a continuous extension $\tilde{f}$ of the original function, where the new function $\tilde{f}$ is the same as f everywhere except at the points of discontinuities, where we define the function to take the value of the limit of f at that point, which exists by definition of a removable discontinuity.

A property of this continuous extension $\tilde{f}$ is that it is a continuous function, and the limit of f at any point equals the value of $\tilde{f}$ at that point.

So another way to think about this is that we are replacing the function $g(t) = \frac{\sin(t)}{t}$ by its continuous extension $\tilde{g}$, and then applying Corollary 2.6 to $\tilde{g} \circ f$.

3 Evaluating Limits

Ok so now I am going to guide you through a process of how to evaluate the limit of a two-variable function (again this all can be generalized to n-variables) such as

$$\lim_{(x,y)\to(a,b)} F(x,y).$$

3.1 Continuous at the point, limit always exists

If $F(x, y)$ is continuous and defined at that point (a, b), you know the limit exists and is the value of the function at that point. So just plug it in. Done.

Example 3.1. Find the limit

$$\lim_{(x,y)\to(a,b)} xy.$$

This is just ab.

Now if the function is not continuous at the point, not defined at the point, or not obvious whether or not it is continuous at the point or not, then we need to do more work.

3.2 Try Factoring or Simplifying

Sometimes it might look like we can factor or simplify the function, then go ahead and do it, often times you will be able to simplify down to see a continuous function that you can then just plug in.

So in particular this often includes polynomials, fractions of polynomials, fractions with radicals, etc.

Example 3.2. Find the limit

$$\lim_{(x,y)\to(1,1)} \frac{(x-y)}{x^2-y^2}.$$

Even though we have polynomials here, but it doesn't seem like we can plug this value in because it will be undefined. But we can rewrite this as

$$\lim_{(x,y)\to(1,1)} \frac{(x-y)}{x^2-y^2} = \lim_{(x,y)\to(1,1)} \frac{(x-y)}{(x-y)(x+y)} = \lim_{(x,y)\to(1,1)} \frac{1}{x+y} = \frac{1}{2}.$$

In the homework, you are asked about

$$\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1}.$$

Try simplifying this to get something that looks nicer.

When you see a fraction with polynomials or radicals, I would immediately start factoring the polynomials if it looks like something you know how to, for example something like differences of squares

$$x^2 - y^2 = (x-y)(x+y).$$

I would also rationalize the radical if it is on the bottom. Again, if these processes seem way to complicated after a few minutes of work, then it is probably not the right direction.

But even if simplifying doesn't actually get you to the point of just plugging in the value, it is often times still worth it to do as it will make any of the following process much easier.

3.3 Maybe the limit doesn't exist

Ok the function is not continuous at the point, you can't plug it in, and it doesn't look like something you can factor or simplify, or even if you had simplified it it is still not obvious what the limit is. Now first thing to do is to suspect that the limit doen not even exist. Because it is often much easier to show that the limit of a multivariable function doesn't exist than it is to show the limit exists. So as we learned in class, the limit can only exist if along every path we take approaching the point, the limit is the same. So the first thing you should do is to try some paths to approach the limit.

I suggest trying the paths $x = 0$, $y = 0$, $x = y$, $y = mx$, $x = my$, $x = y^n$, $y = x^n$, where n might depend on the problem. These are paths that are not hard to try on pretty much any function and can will at most cost you a few minutes to evaluate. If any two of those give you different answers, then you are done since you know that the limit does not exist.

If they all give you the same answer, then you should start to suspect that the limit is that value, and then you now have to go on to try showing that the limit is that value.

Example 3.3. Does the following limit exist?

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}.$$

This is an example from the textbook. If you try the paths $y = mx$, and $x = y^2$, you will get different values, so this limit doesn't exist.

3.4 Can I write the function as the composition of a two-variable function and a single-variable function?

This is what we did in Example 2.9. We all know that the limit of a one-variable limit is much easier to evaluate. For instance, in one-variable, we have the all-powerful L'Hospital's Rule that we can pretty much use to nuke any limit that refuses to just give us the value (by the way, you might be interested to know that L'Hospital didn't actually discover this result, he was some rich guy who paid Johann Bernoulli to do work for him that he would then claim to be his own work).

However, there a few things we need to check before we can proceed. Basically, we want to write $F = g \circ f$ that satisfy certain conditions. First of all, we must know how to evaluate

$$\lim_{(x,y)\to(a,b)} f(x,y) = L.$$

In the example we saw, we had

$$\lim_{(x,y)\to(a,b)} f(x,y) = \lim_{(x,y)\to(a,b)} xy = 0.$$

If we find that this limit actually doesn't exist, then actually the limit of F shouldn't exist either.

If L exists, then we also need to know how to evaluate

$$\lim_{t\to L} g(t).$$

Fortunately, in these types of questions, I feeling is that it would be quite obvious which is the f and which is the g that you need to choose. I think the best way to go about this is to see if you can do some substitution of a variable $t(x, y)$ into F to create a single variable function that you know how to evaluate. But then you have to make sure that this $t(x, y)$ is something you also know how to evaluate as a two-variable function.

3.5 Can I bound the function between two functions?

Example 3.4 (Example 4 in textbook)**.** Find

$$\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2}.$$

After trying out a few paths which all give us the value of 0, we begin to suspect that the limit is 0. This amounts to showing that the distance between the function and 0 gets arbitrarily close as I take $(x, y) \to (0, 0)$. Now the distance between the function and 0 is

$$\left|\frac{3x^2y}{x^2+y^2} - 0\right| = \left|\frac{3x^2y}{x^2+y^2}\right| = \frac{3x^2|y|}{x^2+y^2}.$$

So actually it suffices to show that

$$\lim_{(x,y)\to(0,0)} \frac{3x^2|y|}{x^2+y^2} = 0.$$

Now notice that

$$\frac{x^2}{x^2+y^2} \le 1$$

and thus

$$0 \leq \frac{3x^2|y|}{x^2+y^2} \leq 3|y|.$$

Now by Squeeze Theorem, taking limits we have

$$\lim_{(x,y)\to(0,0)} 0 \leq \lim_{(x,y)\to(0,0)} \frac{3x^2|y|}{x^2+y^2} \leq \lim_{(x,y)\to(0,0)} 3|y|$$

becomes

$$0 \leq \lim_{(x,y)\to(0,0)} \frac{3x^2|y|}{x^2+y^2} \leq 0$$

so

$$\lim_{(x,y)\to(0,0)} \frac{3x^2|y|}{x^2+y^2} = 0.$$

Now, I have to admit that whether or not these bounds are obvious really depends on the person, and to a large extent, your luck because this might be something you've seen, but for someone else this might not be something familiar at all. Thus, as stated before, since evaluating limits is not at all the focus of this course, there won't be any problems like this on the exam.

I have created here a flowchart that you can follow to help you evaluate a limit: