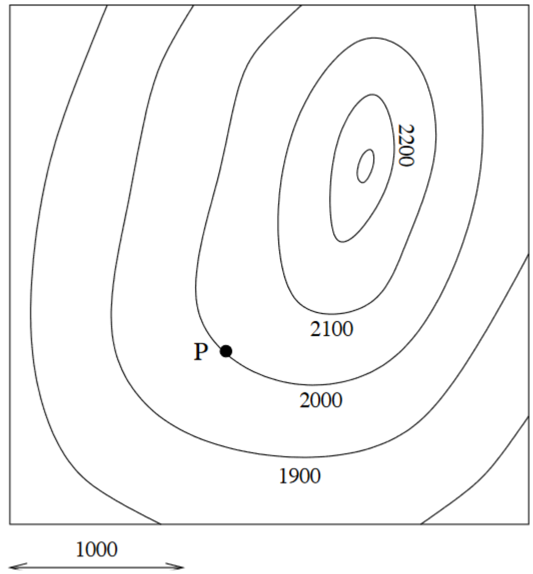
Multivariable Calculus Practice Problems for Midterm 3

November 13, 2020

Note: Starred ($\star$) questions are much harder than questions on the exam. Feel free to skip.

1. Understand intuitively what the boundary of a subset of $\mathbb{R}^n$ means.
2. Understand intuitively what it means for a subset of $\mathbb{R}^n$ to be closed.
3. Understand intuitively what it means for a subset of $\mathbb{R}^n$ to be bounded.
4. Specify whether the following subsets of $\mathbb{R}^n$ are closed, or bounded, both, or neither. Also identify the boundary of each subset.
   (a) $\{1\} \subset \mathbb{R}$.
   (b) The interval $(1, 2) \subset \mathbb{R}$.
   (c) The union of two intervals $(-5, 2] \cup [3, 7] \subset \mathbb{R}$.
   (d) $\{(x, y) \in \mathbb{R}^2 \colon |x - y| < 2\}$.
   (e) $\{(x, y) \colon |(x, y)| < 4\}$
   (f) ($\star$) $\{(x, y) \in \mathbb{R}^2 \colon |x - y| < 2\} \cup \{(x, y) \in \mathbb{R}^2 \colon |x - y| \leq 3\}$.
   (g) ($\star$)$\{(x, y, z) \in \mathbb{R}^3 \colon x^2 + y^2 - z = 0\}$.
   (h) $\{(x_1 \ldots, x_n) \in \mathbb{R}^n \colon x_2 = x_3 = \cdots = x_n = 0\}$.
   (i) ($\star$) $\bigcap_{n=1}^{\infty} \left[-\frac{1}{n}, \frac{1}{n}\right]$.
   (j) ($\star$)$\bigcup_{n=1}^{\infty} [0, n]$.
5. Show that every plane that is tangent to the cone $x^2 + y^2 = z^2$ passes through the origin.
6. Show that every normal line to the sphere $x^2 + y^2 + z^2 = r^2$, for $r > 0$, passes through the center of the sphere. Also show that none of the tangent planes to the sphere passes through the center.
7. On the topographical map below, the level curves for the height function $h(x, y)$ are marked (in feet);

adjacent level curves represent a difference of 100 feet in height.



Mark on the map a point Q at which $h = 2200$, $\frac{\partial h}{\partial x} = 0$, and $\frac{\partial h}{\partial y} < 0$.

8. Given a multivariable function $f(x_1, \ldots, x_n)$, and fix a point $a = (a_1, \ldots, a_n)$,

   (a) Describe in which direction from a the function increases the most rapidly.

   (b) Describe in which direction from a the function decreases the most rapidly.

   (c) Describe in which direction from a the function does not change.

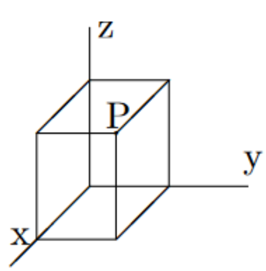
9. State the Extreme Value Theorem. Make sure you understand every part of the statement. Give counterexamples to the claim of the Theorem if each of the assumptions are not met. To be more precise, the EVT assumes continuity of a function, defined on a closed and bounded domain. Give:

   (a) An example of a situation where the claim of the EVT is not met when the function is not continuous.

   (b) An example of a situation where the claim of the EVT is not met when the domain is not closed.

   (c) An example of a situation where the claim of the EVT is not met when the domain is not bounded.

10. A rectangular box in the first octant of $\mathbb{R}^3$ as shown, with one corner at the origin and the three adjacent faces in the coordinate planes. The opposite point P is constrained to lie on the paraboloid $x^2 + y^2 + z = 1$. We are interested in finding a P such that it gives the box the largest volume.



   (a) Show that the problem leads one to maximize $f(x, y) = xy - x^3y - xy^3$, and write down the equations for the critical points of f.

(b) Find a critical point of f which lies in the first quadrant ($x > 0$, $y > 0$).

(c) Determine the nature of this critical point by using the second derivative test.

(d) ($\star$) Find the maximum of f in the first quadrant without using Lagrange multiplier. (You must justify why this point is indeed the maximum)

(e) Solve this problem using Lagrange multiplier.