Multivariable Calculus Practice Problems for Midterm 3

November 13, 2020

Note: Starred ($\star$) questions are much harder than questions on the exam. Feel free to skip.

1. Understand intuitively what the boundary of a subset of $\mathbb{R}^n$ means.
2. Understand intuitively what it means for a subset of $\mathbb{R}^n$ to be closed.
3. Understand intuitively what it means for a subset of $\mathbb{R}^n$ to be bounded.
4. Specify whether the following subsets of $\mathbb{R}^n$ are closed, or bounded, both, or neither. Also identify the boundary of each subset.
   (a) $\{1\} \subset \mathbb{R}$. Closed, bounded, boundary is $\{1\}$
   (b) The interval $(1, 2) \subset \mathbb{R}$. Not closed, bounded, boundary is $\{1, 2\}$.
   (c) The union of two intervals $(-5, 2] \cup [3, 7] \subset \mathbb{R}$. Not closed, bounded, boundary is $\{-5, 2, 3, 7\}$
   (d) ($\star$) $\{(x, y) \in \mathbb{R}^2 \colon |x - y| < 2\}$. Closed, not bounded, boundary is the set itself.
   (e) $\{(x, y) \colon |(x, y)| < 4\}$. This is an open circle of radius 2, so not closed, bounded, and the boundary is $\{(x, y) \colon |(x, y)| = 4\}$.
   (f) ($\star$)$\{(x, y, z) \in \mathbb{R}^3 \colon x^2 + y^2 - z = 0\}$. This is the paraboloid defined by the equation $z = x^2 + y^2$. Closed, not bounded, boundary is the whole set.
   (g) $\{(x_1 \ldots, x_n) \in \mathbb{R}^n \colon x_2 = x_3 = \cdots = x_n = 0\}$. This is the x-axis in $\mathbb{R}^n$, so closed, not bounded, boundary is the whole set itself.
   (h) ($\star$) $\bigcap_{n=1}^{\infty} \left[-\frac{1}{n}, \frac{1}{n}\right]$.
   (i) ($\star$)$\bigcup_{n=1}^{\infty} [0, n]$.

   Here's a useful thing to remember: for a **FINITE** interval, with endpoints a, b that are finite numbers, the interval's boundary are its endpoints.
5. Show that every plane that is tangent to the cone $x^2 + y^2 = z^2$ passes through the origin.

   Solution: Let (x_0, y_0, z_0) be an arbitrary point on the cone. Let

   $$f(x, y, z) = x^2 + y^2 - z^2.$$

   The the cone is the level set of f corresponding to the value 0. We have

   $$\nabla f = \langle 2x, 2y, 1 \rangle$$

   and so

   $$\nabla f(x_0, y_0, z_0) = \langle 2x_0, 2y_0, 1 \rangle.$$

   And this vector should be perpendicular to the tangent plane at (x_0, y_0, z_0), so we can use it as the normal vector for the tangent plane, which has equation

   $$2x_0(x - x_0) + 2y_0(y - y_0) + 2(z - z_0) = 0.$$

We can show that $(0,0,0)$ is a solution to this equation, which would show that the origin lies on the tangent plane:

$$2x_0(0-x_0)+2y_0(0-y_0)+2(0-z_0)=0$$
$$-2x_0^2-2y_0^2-2z_0=0$$
$$x_0^2+y_0^2-z_0=0$$

which we know to be a valid equation because by assumption (x_0,y_0,z_0) lies on the cone, i.e. is a solution to the above equation.

6. Show that every normal line to the sphere $x^2+y^2+z^2=r^2$, for $r>0$, passes through the center of the sphere. Also show that none of the tangent planes to the sphere passes through the center.

   Solution: Let (x_0,y_0,z_0) be an arbitrary point on the sphere. By the same reasoning as that of the previous problem, we set

   $$f(x,y,z)=x^2+y^2+z^2.$$

   Then the sphere corresponds to the level set of f taking value r^2. We find the gradient of f:

   $$\nabla f(x,y,z)=\langle 2x,2y,2z\rangle,$$

   so

   $$\nabla f(x_0,y_0,z_0)=\langle 2x_0,2y_0,2z_0\rangle.$$

   The normal line will be in the direction of this vector because we know that the gradient is perpendicular to the level set (the sphere), so the normal line is

   $$r(t)=\langle x_0,y_0,z_0\rangle+t\langle 2x_0,2y_0,2z_0\rangle$$

   To check that the origin is on this line, we can take $t=\frac{1}{2}$, which would give

   $$r(t)=\langle x_0,y_0,z_0\rangle-\langle x_0,y_0,z_0\rangle=\langle 0,0,0\rangle,$$

   showing that the origin is indeed on the line.

   Using the same gradient vector, we can write down the equation for the tangent plane at (x_0,y_0,z_0):

   $$2x_0(x-x_0)+2y_0(y-y_0)+2z_0(z-z_o)=0.$$

   To show that the origin is not on this plane, we can plug in $(0,0,0)$, and it should give us something non-sense, i.e. $(0,0,0)$ does not solve this equation. So if we plug in $(0,0,0)$, we get

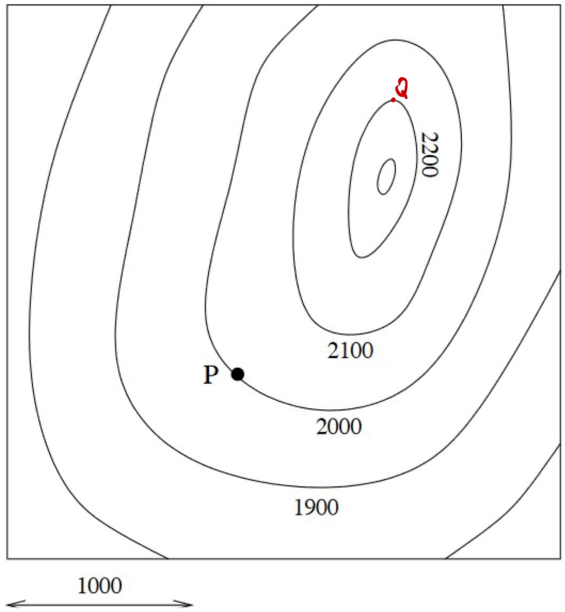
   $$-2x_0^2-2y_0^2-2z_0^2=0$$
   $$x_0^2+y_0^2+z_0^2=0$$

   which is a contradiction because by assumption (x_0,y_0,z_0) lies on the sphere, i.e. it has to satisfy the equation

   $$x_0^2+y_0^2+z_0^2=r^2>0.$$

7. On the topographical map below, the level curves for the height function $h(x,y)$ are marked (in feet);

adjacent level curves represent a difference of 100 feet in height.



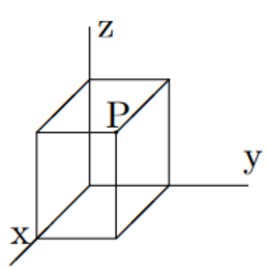
Mark on the map a point Q at which $h = 2200$, $\frac{\partial h}{\partial x} = 0$, and $\frac{\partial h}{\partial y} < 0$.

Firstly, the point has to be on the level curve corresponding to $h = 2200$. We also need that in the x-direction, the function does not change, this amounts to being "flat" in the x-direction, and secondly, we need that in the y-direction, the function decreases. This only gives us one possible place to put Q.

8. Given a multivariable function $f(x_1, \ldots, x_n)$, and fix a point $a = (a_1, \ldots, a_n)$,
   (a) Describe in which direction from a the function increases the most rapidly.
   (b) Describe in which direction from a the function decreases the most rapidly.
   (c) Describe in which direction from a the function does not change.

9. State the Extreme Value Theorem. Make sure you understand every part of the statement. Give counterexamples to the claim of the Theorem if each of the assumptions are not met. To be more precise, the EVT assumes continuity of a function, defined on a closed and bounded domain. Give:
   (a) An example of a situation where the claim of the EVT is not met when the function is not continuous.
   (b) An example of a situation where the claim of the EVT is not met when the domain is not closed.
   (c) An example of a situation where the claim of the EVT is not met when the domain is not bounded.

10. A rectangular box in the first octant of $\mathbb{R}^3$ as shown, with one corner at the origin and the three adjacent faces in the coordinate planes. The opposite point P is constrained to lie on the paraboloid $x^2 + y^2 + z = 1$. We are interested in finding a P such that it gives the box the largest volume.



(a) Show that the problem leads one to maximize $f(x,y) = xy - x^3y - xy^3$, and write down the equations for the critical points of f. *Solution:* If P has coordinates (x,y,z), then the volume is

$$V(x,y,z) = xyz$$

But P is constrained to the paraboloid, which we can write as

$$z = 1 - x^2 - y^2,$$

which if we substitute into V, we get

$$V(x,y,z) = xy(1-x^2-y^2) = xy - x^3y - xy^3.$$

The equations for the critical points are

$$f_x = y - 3yx^2 - y^3 = 0$$

and

$$f_y = x - x^3 - 3xy^2.$$

(b) Find a critical point of f which lies in the first quadrant ($x > 0$, $y > 0$). Using the first derivative test, by solving the above two equations, and imposing that $x, y > 0$, there is only one solution which is

$$x = \frac{1}{2}, y = \frac{1}{2}.$$

(c) Determine the nature of this critical point by using the second derivative test. Using the second derivative test, first by computing

$$f_{xx} = -6yx$$
$$f_{yy} = -6xy$$
$$f_{xy} = f_{yx} = 1 - 3x^2 - 3y^2$$

we get that, at the point $(1/2, 1/2)$:

$$f_{xx}f_{yy} - (f_{xy})^2 > 0$$

and

$$f_{xx} < 0$$

so we have a local minimum.

(d) ($\star$) Find the maximum of f in the first quadrant without using Lagrange multiplier. (You must justify why this point is indeed the maximum)

(e) Solve this problem using Lagrange multiplier. By using Lagrange Multiplier, we solve for the system of equation

$$V(x,y,z) = xyz$$

subject to the constraint

$$g(x,y,z) = x^2 + y^2 + z.$$

First we find gradients

$$\nabla V(x,y,z) = \langle yz, xz, xy\rangle$$
$$\nabla g(x,y,z) = \langle 2x, 2y, 1\rangle$$

So we have the system of equations

$$yz = 2\lambda x$$
$$xz = 2\lambda y$$
$$xy = \lambda$$

Which you can solve to give only one solution if you assume $x, y, z \geq 0$, which is $x = 1/2$, $y = 1/2$, and $z = 1/2$. To actually make sure this is a maximum and not a minimum, observe that you can find values x, y, z that give smaller volume, so this must be a maximum.