Multivariable Calculus Quiz 5 Solution

Due Thursday 10/22 12:00 p.m.

Please show all work to receive full credit.

Don't forget that the timer on Gradescope includes time needed to scan and submit your work!

Problem 1 (3 pts)

Suppose the temperature on a surface is given by the function

$$T(x, y) = 100e^{-(x^2+y^2)}.$$

Now suppose there is a bug that takes the path of a curve given by the function

$$\vec{r}(t) = \langle t\cos(2t), t\sin(2t)\rangle.$$

What is the rate at which the temperature changes as the bug moves?

Solution: Using chain rule on $T(x(t), y(t))$, where $x(t) = tcos(2t)$, and $y(t) = tsin(2t)$, we get

$$\begin{aligned}\frac{\mathrm{d}T}{\mathrm{d}t} &= \frac{\partial T}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial T}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} \\ &= (-200xe^{-(x^2+y^2)})(\cos(2t) - 2t\sin(2t)) + (-200ye^{-(x^2+y^2)})(\sin(2t) + 2t\cos(2t)).\end{aligned}$$

This is enough to receive full credit, although if you continue to simplify, you will get something nice:

$$\frac{\mathrm{d}T}{\mathrm{d}t} = -200te^{-t^2}.$$

Problem 2 (2 pts)

In fluid dynamics, *laminar flow* is characterized by fluid particles following smooth paths in layers, in contrast to something like turbulent flow. Considering laminar flow in a cylinder, the *resistance* R to the flow is related to the length and radius of the cylinder by the *Hagen–Poiseuille law*. More specifically, the resistance is given by the following function:

$$R(w, r) = k\frac{w}{r^4}$$

where w is the length of the cylinder, r is the radius of the cylinder, and k is a constant.

Give the linearization of R at the point (w_0, r_0).

Solution: The linearization is

$$\begin{aligned}L(w, r) &= R(w_0, r_0) + \frac{\partial R(w_0, r_0)}{\partial w}(w - w_0) + \frac{\partial R(w_0, r_0)}{\partial r}(r - r_0) \\ &= k\frac{w_0}{r_0^4} + \frac{k}{r_0^4}(w - w_0) - \frac{4kw_0}{r_0^5}(r - r_0).\end{aligned}$$

Problem 3 (5 pts)

Let $\rho(x, y, z, t)$ be a smooth function (that is, all its partial derivatives exist and are differentiable). Let

$$\tilde{\nabla}\rho = \left\langle \frac{\partial \rho}{\partial x}, \frac{\partial \rho}{\partial y}, \frac{\partial \rho}{\partial z} \right\rangle$$

denote the gradient of ρ **only in the first three variables** (this is not a standard notation, I made this up myself).

Let $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ be a smooth curve, and let $\mathbf{v}(t) = \mathbf{r}'(t)$ denote the velocity of the curve. You can think of $\rho(x, y, z, t)$ as a function that describes the density of a fluid at the point (x, y, z) in space, and at time t. While the curve $\mathbf{r}(t)$ is the path of a point particle in the flow.

For the following questions we restrict to studying the change of ρ on the path $\mathbf{r}(t)$ (otherwise none of the expressions in the questions will make sense).

1. Use the chain rule to show that
$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = \frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \tilde{\nabla}\rho.$$

   Solution: By the chain rule, we have

$$\begin{aligned}
\frac{\mathrm{d}\rho}{\mathrm{d}t} &= \frac{\partial \rho}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial \rho}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial \rho}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t} + \frac{\partial \rho}{\partial t}\frac{\mathrm{d}t}{\mathrm{d}t} \\
&= \left\langle \frac{\partial \rho}{\partial x}, \frac{\partial \rho}{\partial y}, \frac{\partial \rho}{\partial z} \right\rangle \cdot \left\langle \frac{\mathrm{d}x}{\mathrm{d}t}, \frac{\mathrm{d}y}{\mathrm{d}t}, \frac{\mathrm{d}z}{\mathrm{d}t} \right\rangle + \frac{\partial \rho}{\partial t} \\
&= \frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \tilde{\nabla}\rho.
\end{aligned}$$

2. We say that the fluid is *incompressible* if
$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = 0.$$

   (a) Suppose that the density function ρ is independent of the spatial variables x, y, z, i.e. $\rho(x, y, z, t) = \rho(t)$. Use the result of Part 1. to give the condition on ρ for the fluid to be incompressible.
   Solution: If $\rho(x, y, z, t) = \rho(t)$, then $\tilde{\nabla}\rho = 0$, so $\mathbf{v} \cdot \tilde{\nabla}\rho = 0$. Therefore
$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = 0$$
   if and only if
$$\frac{\partial \rho}{\partial t} = 0.$$

   (b) Now suppose instead that the density function ρ is **only dependent on the spatial variables** x, y, z, i.e. $\rho(x, y, z, t) = \rho(x, y, z)$. In this case if a flow is incompresible, we call it a *stratified flow*. Use the result of Part 1. to give the condition for ρ and $\mathbf{v}$ for a stratified flow. Note: Say something about the relation between ρ and $\mathbf{v}$.
   Solution: If $\rho(x, y, z, t) = \rho(x, y, z)$, then $\frac{\partial \rho}{\partial t} = 0$, thus
$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = \mathbf{v} \cdot \tilde{\nabla}\rho.$$
   So the fluid is incompresible if and only if
$$0 = \mathbf{v} \cdot \tilde{\nabla}\rho$$
   i.e. $\mathbf{v}$ and $\tilde{\nabla}$ are perpendicular.