Multivariable Calculus Quiz 6 Solution

Due Thursday 11/5 12:00 p.m.

Please show all work to receive full credit.
Don't forget that the timer on Gradescope includes time needed to scan and submit your work!
Time limit: 37 minutes

Problem 1 (4 pts)

Find parametric equations for the tangent line to the curve of intersection of the paraboloid $z = x^2 + y^2$ and the ellipsoid $4x^2 + y^2 + z^2 = 9$ at the point $(-1, 1, 2)$. (Hint: it is not necessary to actually find the curve of intersection)

Solution: The tangent line to the curve of intersection must be perpendicular to the normal vector of both surfaces at the point. Thus we first find the two normal vectors at the point, take their cross product, then the resulting vector should be a vector in the direction of the tangent line. Let us define

$$f(x, y, z) = x^2 + y^2 - z$$

and

$$g(x, y, z) = 4x^2 + y^2 + z^2.$$

Then

$$\nabla f(x, y, z) = \langle 2x, 2y, -1 \rangle$$

so

$$\nabla f(-1, 1, 2) = \langle -2, 2, -1 \rangle.$$

$$\nabla g(x, y, z) = \langle 8x, 2y, 2z \rangle$$

so

$$\nabla g(-1, 1, 2) = \langle -8, 2, 4 \rangle.$$

Taking cross products:

$$\begin{vmatrix} i & j & k \\ -2 & 2 & -1 \\ -8 & 2 & 4 \end{vmatrix} = \langle 6, 0, 12 \rangle.$$

Thus the tangent line to the curve of intersection is

$$r(t) = \langle -1, 1, 2 \rangle + t\langle 6, 0, 12 \rangle = \langle -1 + 6t, 1, 12t \rangle$$

with parametric equations

$$x(t) = 1 + 6t, y(t) = 1, z(t) = 12t.$$

Problem 2 (2 pts)

Consider the function $T(x, y, z) = x^2 + 2y^2 + 2z^2$, and let P be the point $(1, 1, 1)$. In which directions (there are a lot) from P does the value of the function not change?

Solution: We aim to find $\hat{u}$ such that

$$D_{\hat{u}}T(1, 1, 1) = \nabla T(1, 1, 1) \cdot \hat{u} = 0.$$

This is equivalent to

$$\langle 2, 4, 4 \rangle \cdot \hat{u} = 0.$$

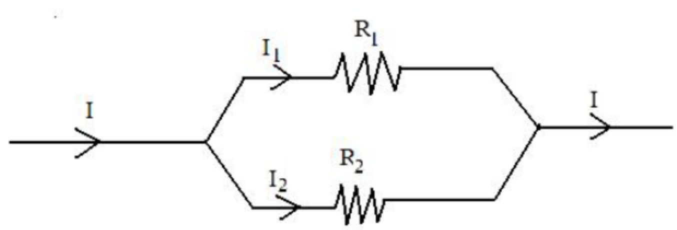
Thus in the direction of any vector that is perpendicular to $\langle 2, 4, 4 \rangle$, the function does not change.

Problem 3 (4 pts)

An electric current I through a resistor with resistance R results in *energy loss* given by

$$\text{energy loss} = I^2 R.$$

Suppose we have the following situation:



where I is split into two currents I_1 and I_2 (and so they add up to the total current I), flowing through resistors with resistance R_1 and R_2 respectively. Determine what choice of I_1 and I_2 will minimize the total energy loss. *Solution:* We want to minimize the function

$$f(I_1, I_2) = I_1^2 R_1 + I_2^2 R_2$$

subject to the constraint

$$g(I_1, I_2) = I_1 + I_2 = I.$$

Using Lagrange Multipliers, this amounts to solving the system of equations:

$$\nabla f = \lambda \nabla g$$

which is

$$\langle 2I_1R_1, 2I_2R_2 \rangle = \lambda \langle 1, 1 \rangle.$$

Which becomes

$$\begin{cases} 2I_1R_1 = \lambda \\ 2I_2R_2 = \lambda \\ I_1 + I_2 = I \end{cases}$$

Solving this gives

$$I_1 = \frac{IR_2}{R_1 + R_2}, \quad I_2 = \frac{IR_1}{R_1 + R_2}.$$