Practice Final

Due Tuesday 12/14 9:00 a.m.

Everything is extra credit, so there is no possibility of you losing credit towards your total quiz grade for the course. In other words this is positively graded so the more correct work you show the more extra credit you get, while incorrect work is not penalized but just not counted towards the extra credit you will receive. The maximum amount of extra credit you can possibly receive is equivalent to that of 1.5 quizzes.

Time limit: None

Problem 1

Find the angle between the vectors $\langle 1,1,2\rangle$ and $\langle 13,2,11\rangle$.

Solution:

$$\cos\theta = \frac{\langle 1,1,2\rangle \cdot \langle 13,2,11\rangle}{|\langle 1,1,2\rangle||\langle 13,2,11\rangle|} = \frac{37}{\sqrt{6}\cdot 7\sqrt{6}} = \frac{37}{42}$$

thus

$$\theta = \cos^{-1}\left(\frac{37}{42}\right).$$

Problem 2

Find the volume of the pralellopiped spanned by the vectors $\langle 1,0,0\rangle, \langle 1,1,0\rangle$, and $\langle 1,1,2\rangle$.

Solution:

$$V = (\langle 1,0,0\rangle \times \langle 1,1,0\rangle)\cdot\langle 1,1,2\rangle = \langle 0,0,1\rangle \cdot \langle 1,1,2\rangle = 2.$$

Problem 3

Given the points $P=(1,1,-1), Q=(1,2,0)$, $R=(-2,2,2)$,

1. Find a vector perpendicular to both PQ and PR.

   Solution: PQ is the vector $\langle 1,2,0\rangle - \langle 1,1,-1\rangle = \langle 0,1,1\rangle$ and PR is the vector $\langle -2,2,2\rangle - \langle 1,1,-1\rangle = \langle -3,1,3\rangle$. We can take the cross product to get a vector perpendicular to both:

   $$\langle 0,1,1\rangle \times \langle -3,1,3\rangle = \langle 2,-3,3\rangle.$$

2. Find the equation of a plane passing through P, Q, and R. *Solution:* We can use the vector we found as the normal vector, and one of the three points, for instance if we use the point $P=(1,1,-1)$:

   $$2(x-1)-3(y-1)+3(z+1)=0.$$

Problem 4

Find the parametric equations of a line that passes through $(1,1,1)$ and is perpendicular to the unit sphere in $\mathbb{R}^3$.

Take the line that passes through $(1,1,1)$ and the origin.

Problem 5

Let L denote the line which passes through $(0,0,1)$ and is parallel to the line in the xy-plane given by $y = 2x$.

1. Give the parametric equations and symmetric equations of L.

   Solution: The direction vector can be taken to be $\langle 1,2,0\rangle$ because the line $y = 2x$ has the direction vector $\langle 1,2,0\rangle$, and we want a line that is parallel to this, i.e. have the same direction vector. We have a point $P_0 = (0,0,1)$ so the equation for L is

   $$r(t) = \langle 0,0,1\rangle + t\langle 1,2,0\rangle.$$

   i.e. parametric equations are

   $$x = t, y = 2t, z = 1$$

   and symmetric equations are

   $$x = \frac{y}{2}, z = 1.$$

2. Let $\mathcal{P}$ be the plane which passes through $(0,0,1)$ and is *perpendicular* to the line L. Give the equation of $\mathcal{P}$.

   Since $\mathcal{P}$ must be perpendicular to L, we can take the normal vector of $\mathcal{P}$ to be the direction vector of L, i.e. $\langle 1,2,0\rangle$. So the equation for $\mathcal{P}$ is

   $$1(x-0) + 2(y-0) + 0(z-1)$$

   or

   $$x + 2y = 0.$$

3. Suppose that the point $Q \neq (0,0,1)$ lies on L. Write down the method or formula you would use to find the point Q^* which is (all of the following must hold)

   (a) on L;

   (b) the same distance away from the point $(0,0,1)$ as Q;

   (c) and is on the *other* side of $\mathcal{P}$ from Q.

Solution: Q being on L means that $Q = (t, 2t, 1)$ for some $t \neq 0$. Now on the line L, $t = 0$ corresponds to the point $(0,0,1)$ so if we reverse the signs of the first two coordinates of Q (the two coordinates that depend on t) we would get a point that is of the same distance from $(0,0,1)$ but traversed in the opposite direction from $(0,0,1)$, giving us the desired Q^*:

$$Q^* = (-t, -2t, 1).$$

Problem 6

The position vector of a point P on a curve C is $r(t) = \langle 3\cos t, 5\sin t, 4\cos t\rangle$.

1. Show that the speed at the point is constant.

   Solution: The first derivative is

   $$r'(t) = \langle -3\sin t, 5c\cos t, -4\sin t\rangle$$

   which has magnitude

   $$|r'| = \sqrt{9\sin^2 t + 16\sin^2 t + 25\cos^2 t} = 5.$$

2. At what point $A = (a,b,c)$ does the curve pass through the yz-plane?

   Solution: Passes through yz-plane when $x = 0$ because when $\cos t = 0$ we must have $t = \frac{\pi}{2}, \frac{3\pi}{2}$, which correspond to the points $(0, \pm 5, 0)$.

Problem 7

Let C be the curve on the xy-plane defined by the parabola $y = x^2$. Compute the curvature of C at every point on C.

Solution: We can use the parametrization

$$r(t) = \langle t, t^2 \rangle \quad t \in \mathbb{R}.$$

which has first derivative

$$r'(t) = \langle 1, 2t \rangle.$$

normalizing:

$$T(t) = \frac{\langle 1, 2t \rangle}{\sqrt{1+4t^2}}.$$

which has first derivative

$$T'(t) = \langle \rangle$$

$$|T'| = \left\langle -\frac{4t}{(1+4t^2)^{3/2}}, \frac{2}{(1+4t)^{3/2}} \right\rangle$$

which has magnitude

$$|T'(t)| = \sqrt{\left(\frac{16x^2}{(4x^2+1)^3}\right) + \frac{4}{(4x+1)^3}}$$

so the curvature is

$$\kappa(t) = \frac{|T'(t)|}{|r'(t)|} = \frac{\sqrt{\left(\frac{16x^2}{(4x^2+1)^3}\right) + \frac{4}{(4x+1)^3}}}{\sqrt{1+4t^2}}$$

Problem 8

Let

$$u = x^2 + yz, \quad x = pr\cos\theta, \quad y = pr\sin\theta, \quad z = p + r.$$

Find $\frac{\partial u}{\partial p}, \frac{\partial u}{\partial r}, \frac{\partial u}{\partial \theta}$ when $p = 2, r = 3, \theta = 0$.

Solution:

$$\begin{aligned}
\frac{\partial u}{\partial p} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial p} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial p} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial p} \\
&= (2x)(r\cos\theta) + (z)(r\sin\theta) + (y)(1) \\
&= 2pr^2\cos^2\theta + (p+r)r\sin\theta + pr\sin\theta
\end{aligned}$$

so at $p = 2, r = 3, \theta = 0$:

$$\frac{\partial u}{\partial p}(p = 2, r = 3, \theta = 0) = 36.$$

$$\begin{aligned}
\frac{\partial u}{\partial r} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial r} \\
&= (2x)(p\cos\theta) + (z)(p\sin\theta) + (y)(1) \\
&= 2rp^2\cos^2\theta + (p+r)(p\sin\theta) + pr\sin\theta
\end{aligned}$$

so at $p = 2, r = 3, \theta = 0$:

$$\frac{\partial u}{\partial r}(p = 2, r = 3, \theta = 0) = 24.$$

$$\begin{aligned}\frac{\partial u}{\partial \theta} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial \theta} \\ &= (2x)(-pr\sin\theta) + (z)(pr\cos\theta) \\ &= -2p^2r^2\sin^2\theta\cos^2\theta + (p+r)(pr\cos\theta)\end{aligned}$$

so at $p = 2, r = 3, \theta = 0$:

$$\frac{\partial u}{\partial \theta}(p = 2, r = 3, \theta = 0) = 30.$$

Problem 9

Let $F(x, y) = x^2y - xy^3$, and $P = (2, 1)$. Find the directional derivative of F at P in the direction $\langle \frac{3}{5}, \frac{4}{5} \rangle$.

Solution: Since

$$\nabla F = \langle 2xy - y^3, x^2 - 3xy^2 \rangle$$

we have

$$\begin{aligned} D_uF(2, 1) &= \nabla F(2, 1) \cdot \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle \\ &= \langle 3, -2 \rangle \cdot \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle \\ &= \frac{1}{5} \end{aligned}$$

Problem 10

Find the point on the plane $2x + y - z = 6$ which is closest to the origin using Lagrange multipliers.

Solution: We want to minimize $f(x, y, z) = x^2+y^2+z^2$ subject to the constraint $g(x, y, z) = 2x+y-z = 6$. First we compute

$$\nabla f(x, y, z) = \langle 2x, 2y, 2z \rangle$$

$$\nabla g(x, y, z) = \langle 2, 1, -1 \rangle$$

Using Lagrange multiplier, we solve for the system of equations:

$$\begin{cases} 2x = 2\lambda \\ 2y = \lambda \\ 2z = -\lambda \\ 2x + y - z = 6 \end{cases}$$

which has solution

$$x = 2, y = 1, z = -1, \lambda = 2.$$

So there is one extremal point $(2, 1, -1)$, which gives $x^2 + y^2 + z^2 = 6$ we just need to make sure this is a minimum, but we can easily find a point that gives a larger value on the curve, for instance $(3, 0, 0)$ gives $x^2 + y^2 + z^2 = 9$. So the point we found is a minimum.

Problem 11

1. Write down the change of coordinate formulas for spherical coordinates, i.e. write x, y, and z in terms of the spherical coordinates.

   Solution:

$$x = \rho\sin\phi\cos\theta, \quad y = \rho\sin\phi\sin\theta, \quad z = \rho\cos\phi.$$

2. Using spherical coordinates, describe the region in $\mathbb{R}^3$ in the first octant that is bounded by the sphere centered at the origin with radius 4 .

   Solution:
$$\{0 \leq \phi \leq \frac{\pi}{2}, 0 \leq \theta \leq \frac{\pi}{2}, 0 \leq \rho \leq 4\}.$$

Problem 12

Consider the following domains in $\mathbb{R}^2$:

$$D_1 = \{(x, y) \mid y = x^2\}$$

$$D_2 = \{(x, y) \mid 1 \leq x^2 + y^2 < 2\}$$

Which domains above is/are simply connected? D_1 is simply connected, D_2 is not.

Which domains above is/are closed? D_1 is closed, D_2 is not.

Which domains above is/are bounded? D_2 is bounded, D_1 is not.

Problem 13

Find the Jacobian corresponding to the change-of-variables from Cartesian coordinates to polar coordinates on the plane.

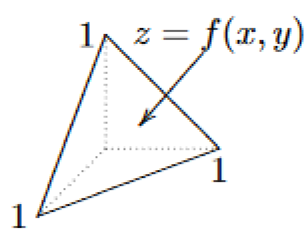
Solution: We know that the Jacobian should be r, we can verify this. by computing

$$\begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial r} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial r} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial r} \end{vmatrix}.$$

Details omitted.

Problem 14

Find the volume of the tetrahedron shown below.



Tetrahedron

Solution: We want to integrate the plane $x + y + z = 1$,over the region which is a triangle described by

$$\{(x, y) \mid 0 \leq x \leq 1, 0 \leq y \leq -x + 1\}$$

so the volume is

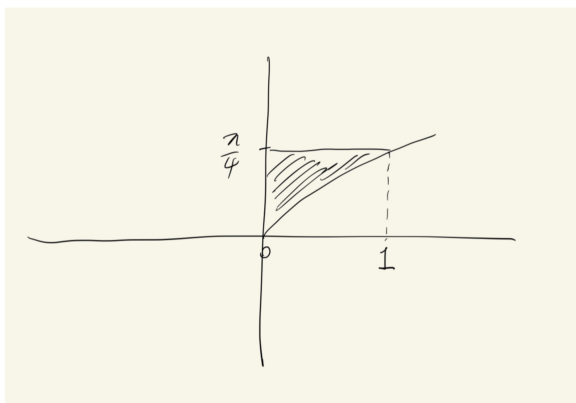
$$\int_0^1 \int_0^{-x+1} 1 - x - y \mathrm{d}y\mathrm{d}x = \int_0^1 \left[y - xy - \frac{1}{2}y^2 \right]_0^{-x+1} \mathrm{d}x$$
$$= \int_0^1 \frac{1}{2}x^2 - x + \frac{1}{2}\mathrm{d}x$$
$$= \left[\frac{1}{6}x^3 - \frac{1}{2}x^2 + \frac{1}{2}x \right]_0^1$$
$$= \frac{1}{6}$$

Problem 15

Sketch the region of integration and change the order of integration.

$$\int_0^1 \int_{\arctan x}^{\pi/4} f(x, y)\mathrm{d}y\mathrm{d}x$$

Solution:



This is written as a type I region, so we will change it to be a type II region:

$$\{(x, y) \mid 0 \leq \pi/4, 0 \leq x \leq \tan(y)\}$$

to get

$$\int_0^{\pi/4} \int_0^{\tan y} f(x, y)\mathrm{d}x\mathrm{d}y.$$

Problem 16

Find the volume of the solid bounded by the following: inside the sphere $x^2 + y^2 + z^2 = 16$ and outside the cylinder $x^2 + y^2 = 4$.

Solution: We use cylindrical coordinates.

$$
\begin{aligned}
V &= \int_0^{2\pi}\int_2^4\int_{-\sqrt{16-r^2}}^{\sqrt{16-r^2}} r\mathrm{d}z\mathrm{d}r\mathrm{d}\theta \\
&= \int_0^{2\pi}\int_2^4 2r\sqrt{16-r^2}\mathrm{d}r\mathrm{d}\theta \\
&= \int_0^{2\pi}\left[\frac{-2}{3}(16-r^2)^{3/2}\right]_{r=2}^{r=4}\mathrm{d}\theta \\
&= \int_0^{2\pi}\frac{2}{3}(12)^{3/2}\mathrm{d}\theta \\
&= \frac{4\pi}{3}(12)^{3/2}
\end{aligned}
$$

Problem 17

Evaluate

$$\int_C (y-x)\mathrm{d}x + e^x\mathrm{d}y$$

where C is the line segment from $P=(1,1)$ to $Q=(2,4)$.

Solution: We can take the parametrization of the line segment to be

$$r(t) = (1-t)\langle 1,1\rangle + t\langle 2,4\rangle = \langle 1+t, 1+3t\rangle.$$

So

$$x(t) = 1+t, \quad x'(t) = 1.$$

$$y(t) = 1+3t, \quad y'(t) = 3.$$

Then

$$\int_C (y-x)\mathrm{d}x = \int_0^1 ((1+3t)-(1+t))\mathrm{d}t = \int_0^1 2t\mathrm{d}t = 1.$$

And

$$\int_C e^x 3\mathrm{d}y = 3\int_0^1 e^{1+t}\mathrm{d}t = [e^{1+x}]_0^1 = 3e^2 - 3e.$$

So

$$\int_C (y-x)\mathrm{d}x + e^x\mathrm{d}y = 1 + 3e^2 - 3e.$$

Problem 18

Evaluate the line integral

$$\int \mathbf{F}\cdot\mathrm{d}\mathbf{r}$$

where

$$\mathbf{F}(x,y,z) = \sin x\mathbf{i} + \cos y\mathbf{j} + xz\mathbf{k}$$

and

$$\mathbf{r}(t) = t^3\mathbf{i} - t^2\mathbf{j} + t\mathbf{k}, \quad 0 \le t \le 1.$$

Solution:

$$\begin{aligned}
\int \mathbf{F}\cdot \mathrm{d}\mathbf{r} &= \int_0^1 \langle \sin t^3, \cos t^2, t^4\rangle \cdot \langle 3t^2, -2t, 1\rangle \mathrm{d}t \\
&= \int_0^1 (3t^2 \sin t^3 - 2t\cos t^2 + t^4)\mathrm{d}t \\
&= \left[\frac{1}{5}\big(t^5 - 5\cos\big(t^3\big) - 5\sin\big(t^2\big)\big)\right]_0^1 \\
&= \text{this is enough to get credit.}
\end{aligned}$$

Problem 19

Define what a conservative vector field is. A vector field is conservative if it is the gradient of a scalar valued function.

Problem 20

Let

$$\mathbf{F} = ay^2\mathbf{i} + 2y(x+z)\mathbf{j} + (by^2 + z^2)\mathbf{k}$$

be a vector field.

1. For what values of a and b will $\mathbf{F}$ be conservative?
2. Using these values, find a function $f(x, y, z)$ such that $\mathbf{F} = \nabla f$.
3. Using these values, give the equation of a surface S having the property

$$\int_C \mathbf{F}\cdot \mathrm{d}\mathbf{r} = 0$$

for a curve C from the point P to Q on the surface.

solution omitted.

Problem 21

Evaluate the following integral directly, then evaluate it using Green's Theorem:

$$\oint_C (x-y)\mathrm{d}x + (x+y)\mathrm{d}y$$

where C is the circle centered at the origin with radius 2. By Green's Theorem,

$$\begin{aligned}
\oint_C (x-y)\mathrm{d}x + (x+y)\mathrm{d}y &= \int\int_D (1+1)\,\mathrm{d}A \\
&= \int_0^{2\pi}\int_0^2 2r\mathrm{d}r\mathrm{d}\theta \\
&= \int_0^{2\pi} 4\mathrm{d}\theta \\
&= 8\pi.
\end{aligned}$$

Problem 22

Use Green's Theorem to evaluate $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$, where

$$F(x,y) = \langle \sqrt{x} + y^3, x^2 + \sqrt{y} \rangle$$

and where C consists of the arc of the curve $y = \sin x$ from $(0,0)$ to $(\pi, 0)$ and the line segment from $(\pi, 0)$ to $(0,0)$.

Solution: We have

$$\frac{\partial Q}{\partial x} = 2x, \quad \frac{\partial P}{\partial y} = 3y^2.$$

Then

$$\begin{aligned} \int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} &= \int_\pi^0 \int_0^{\sin x} 2x - 3y^2 \mathrm{d}y\mathrm{d}x \\ &= \frac{4}{3} - 2\pi. \end{aligned}$$

Note that there we have x bounds from π to 0 because we traverse from $(\pi, 0)$ to $(0,0)$.

Problem 23

Compute $\int_C \mathbf{F}\mathrm{d}\mathbf{r}$, where

$$\mathbf{F}(x,y) = \frac{2xy\mathbf{i} + (y^2 - x^2)\mathbf{j}}{(x^2+y^2)^2}$$

and C is any positively oriented simple closed curve that encloses the origin.

Solution: We will make use of the "general version" of Green's Theorem. We will take C' to be a circle centered at the origin small enough such that it sits within C, with counterclockwise orientation. Then the Theorem gives

$$\begin{aligned} \int_C P\mathrm{d}x + Q\mathrm{d}y + \int_{-C'} P\mathrm{d}x + Q\mathrm{d}y &= \int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathrm{d}A \\ &= \int\int_D \left[\frac{2x(x^2-3y^2)}{(x^2+y^2)^3} - \frac{2x(x^2-3y^2)}{(x^2+y^2)^3} \right] = 0 \end{aligned}$$

where D is the region enclosed by C. If we let a denote the radius of C, and so C has the parametrization

$$r(t) = \langle a\cos t, a \sin t \rangle, \quad 0 \le t \le 2\pi$$

which has first derivative

$$r'(t) = \langle -a\sin t, a\cos t \rangle$$

then we have

$$\begin{aligned} \int_C P\mathrm{d}x + Q\mathrm{d}y &= \int_{C'} P\mathrm{d}x + Q\mathrm{d}y \\ &= \int_0^{2\pi} \left\langle \frac{2(a\cos t)(a \sin t)}{((a^2\cos^2 t) + a^2 \sin^2 t)^2}, \frac{a^2\sin^2 t - a^2\cos^2 t}{(a^2\cos^2 t + a^2\sin^2 t)^2} \right\rangle \cdot \langle -a\sin t, a\cos t \rangle \mathrm{d}t \\ &= \int_0^{2\pi} \frac{-2a\sin t(a\cos t)(a\sin t) + (a\cos t)a^2\sin^2 t - a^2\cos^2 t}{(a^2\cos^2 t + a^2\sin^2 t)^2} \mathrm{d}t \end{aligned}$$

yea at this point I really don't want to do this integral, but you get the idea, this is enough to get full credit on this practice exam.