The Gromov Norm

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1 Introduction

Let M be a connected closed oriented n-manifold.

1.1 Fundamental Class

Recall that $H_n(M;\mathbb{Z}) \cong \mathbb{Z}$ has a preferred generator denoted by $[M]$, called the **fundamental class** of M. To obtain a cycle representing $[M]$, pick some Δ-complex (or triangulation) of M. Then, $[M]$ is represented by the sum of the n-simplices in this Δ-structure (being careful to properly orientate these simplices).

We want to work with homology with real coefficients, so by the standard map $H_n(M;\mathbb{Z}) \to H_n(M;\mathbb{R})$ induced by the inclusion $\mathbb{Z} \to \mathbb{R}$, we can similarly view $[M]$ as a preferred generator for $H_n(M;\mathbb{R}) \cong \mathbb{R}$.

1.2 The Simplicial and Gromov Norms

Let $C_k(M,\mathbb{R})$ be the set of k-chains in M with coefficients in $\mathbb{R}$. For $c \in C_k(M,\mathbb{R})$, write $c = \sum_{i=1}^{p} a_i\sigma_i$ where $a_i \in \mathbb{R}$ and σ_i a singular k-simplex $\forall i$, with $\sigma_i \neq \sigma_j$ for $i \neq j$. Then, we define the **simplicial norm**

$$\|c\| = \sum_{i=1}^{p} |a_i|.$$

In some sense, this is counting the number of simplices it takes to write c, where we're of course allowing fractional simplices.

Now, we define the **Gromov norm** of M to be

$$\|M\| = \inf\{\|c\| : c \in C_n(M;\mathbb{R}), c \text{ a cycle }, [c] = [M]\}.$$

Similar to above, we can think of $\|M\|$ as measuring the infimal number of simplices it takes to represent $[M]$, where again we're allowing fractional simplices. As we'll see, there are non-trivial manifolds for which the Gromov norm is zero, so for a number of reasons this is not a true norm.

Remark: One might wonder if one will be able to find a Δ-structure for M consisting of $\|M\|$ n-simplices. This will not be the case in the examples we'll consider.

2 Computing with Σ_g

We'll do some computations related to the Gromov norm, motivated by trying to determine $\|\Sigma_g\|$, where Σ_g is the closed orientable surface of genus g.

Example 1. We'd like to consider $\|\Sigma_g\|$, where Σ_g is the closed orientable surface of genus g. Recall that we have a standard Δ-structure for Σ_g with $4g-2$ 2-simplices (obtained from a $4g$-gon). As mentioned above, the chain given by the sum of these $4g-2$ simplices (properly oriented) gives a representative of $[\Sigma_g]$. Thus, we have $\|\Sigma_g\| \leq \sum_{i=1}^{4g-2} |1| = 4g-2$. $\square$

We'll try to do some improvements on this bound.

Proposition 1. If $p : M \to N$ is a degree $d < \infty$ covering map of oriented closed connected manifold, then $\|M\| = d\|N\|$.

Proof. Let $c = \sum a_i \sigma_i$ be a cycle representing $[M]$. Then, $p \circ c = \sum a_i(p \circ \sigma_i)$ is a cycle representing $p_*([M]) = \pm d[N]$. Thus, $\sum(\pm a_i/d)(p \circ \sigma_i)$ represents $[N]$, so we see

$$\|N\| \leq \sum(|a_i|/d) = (1/d)\sum |a_i|.$$

Taking the infimum over the right hand side, as in the definition of $\|M\|$, we have $d\|N\| \leq \|M\|$.

To establish the reverse inequality, let $s = \sum b_i \tau_i$ be a cycle representing $[N]$. Since Δ^n is simply connected, each map $\tau_i : \Delta^n \to N$ has d lifts $\tilde{\tau}_i^1, ..., \tilde{\tau}_i^d$ to M. Then, $\tilde{s} := \sum_i (b_i \sum_{j=1}^d \tilde{\tau}_i^j)$ is a cycle in M, and $p_*([\tilde{s}]) = d[s] = d[N] = \pm p_*([M])$. Since p_* is injective (being a non-zero linear map between rank 1 vector spaces), we have $[\pm\tilde{s}] = [M]$. Thus, we can conclude that $\|M\| \leq \sum_i d|b_i| = d\sum_i |b_i|$. Taking the infimum over the right hand side then yields $\|M\| \leq d\|N\|$. $\square$

Example 2. Let's consider the torus $T = \Sigma_1$. Recall that for any $d \in \mathbb{N}$, there is a d-sheeted covering map $T \to T$ (by winding around itself d times). Thus, Proposition 1 tells us that for all $d \in \mathbb{N}$, we have $\|T\| = d\|T\|$, so we must have $\|T\| = 0$. $\square$

We can also use the proposition to improve upon Example 1:

Example 3. Consider Σ_g, for $g \geq 2$. Recall (eg. Hatcher pg 73) that for all $d \in \mathbb{N}$, there is a d-sheeted covering $\Sigma_{d(g-1)+1} \to \Sigma_g$. Thus, we have

$$d\|\Sigma_g\| = \|\Sigma_{d(g-1)+1}\| \leq 4(d(g-1)+1) - 2 = 4d(g-1) + 2$$

where the first equality comes from Proposition 1 and the inequality comes from Example 1. Hence, dividing both sides by d gives us $\|\Sigma_g\| \leq 4(g-1) + (2/d)$. Then, letting $d \to \infty$, we obtain $\|\Sigma_g\| \leq 4(g-1)$. $\square$

Fact. For $g \geq 2$, $\|\Sigma_g\| = 4(g-1)$.

This fact is not too deep, but does take a little bit of machinery (eg. hyperbolic geometry) to prove (see Benedetti and Petronio B.3.3, C.2.3, C.4.6).

Above, it was suggested that the Gromov norm gives a way of measuring the number of simplices it takes to represent the fundamental class. One may wonder, then, if there is a Δ-complex structure for Σ_g consisting of $4(g-1)$ 2-simplices. We'll show that this is not the case:

Example 4. Suppose (for contradiction) we have a Δ-complex structure for Σ_g consisting of $4(g-1)$ 2-simplices. There are then $(3/2)4(g-1)$ edges in this decomposition, so if v is the number of vertices, we have

$$2 - 2g = \chi(\Sigma_g) = 4(g-1) - (3/2)4(g-1) + v = 2 - 2g + v,$$

so we must have $v = 0$, which cannot be the case. $\square$

Remark: One can show that for any closed orientable hyperbolic n-manifold, the infimum can never by achieved, even with fractional coefficients. (Proof: As in Prop C.4.6 in BP, we have $\mathrm{vol}(M) = \sum a_i \cdot \mathrm{algvol}(\sigma_i) = \sum |a_i|\mathrm{sgn}(a_i)\mathrm{algvol}(\sigma_i) < v_n \sum |a_i|$; then apply C.4.2).

Applications to Hyperbolic Geometry

An n-simplex in $\overline{\mathbb{H}^n}$ with hyperbolic faces (and in particular geodesic edges) is said to be **ideal** if all of its vertices lie in $\partial\mathbb{H}^n$ and is said to be **regular** if (somewhat informally) it has maximal symmetry with respect to isometries of $\mathbb{H}^n$. Now, let v_n be the volume of a regular ideal n-simplex. Note that, in fact, all regular ideal n-simplices have the same volume, so this definition makes sense.

Theorem 1. If M is an oriented compact hyperbolic n-manifold, then $\|M\| = \mathrm{vol}(M)/v_n$.

Let us remark that since $\|M\|$ is defined purely topologically, this in particular tells us that the hyperbolic volume of M is a topological (and in fact a homotopy) invariant, which is somewhat surprising.

This suggests a connection between the topology and geometry of hyperbolic manifolds, and that the Gromov norm may be useful in demonstrating these connections. Indeed, when $n \geq 3$, the topology of a hyperbolic n-manifold determines the geometry in a very strong way, as the next theorem states. A proof of this theorem was given by Gromov and Thurston in 1982 using the Gromov norm.

Theorem 2. (Mostow Rigidity) If M_1, M_2 are compact oriented hyperbolic n-manifolds with $n \geq 3$ such that $\pi_1(M_1) \cong \pi_1(M_2)$, then M_1 and M_2 are isometric.